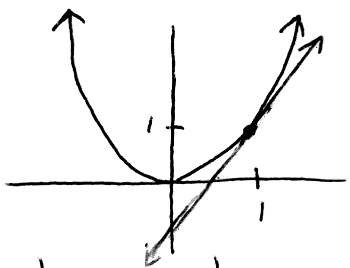
Unit 2 | Derivatives & Differentiation

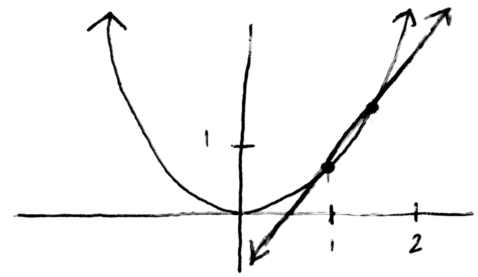
I. Derivatives and Tangent Lines (2A, 2B)

A. Objective: Find the slope of a tangent line on a curve at a point

$y=x^2$



tangent



secant

Slope $= m = \dfrac{y_2-y_1}{x_2-x_1}$ but...

we only have 1 point so choose a point on the curve that is close by

$x=1.5 \quad y=(1.5)^2=2.25$

$(1.5, 2.25)$ & $(1,1)$

$m = \dfrac{2.25-1}{1.5-1} = \dfrac{1.25}{.5} = 2.5$

$1 = 2.5(1)+b$

$-1.5 = b$

$\boxed{y=2.5x-1.5}$

estimate

But! The closer the chosen point is to the given point, the better the approximation gets.

In general, the equation for the slope of the secant line between $(x, f(x))$ and $(a, f(a))$ is:

$$m_{sec} = \frac{f(x)-f(a)}{x-a}$$

Also written as:

$$m_{sec} = \frac{f(a+h)-f(a)}{a+h-a}$$

$$m_{sec} = \frac{f(a+h)-f(a)}{h}$$

where "h" represents some very small distance away from "a" on the x-axis.

If we want the x-value to get closer and closer to "a", that means "h" has to get closer and clocer to 0.

So $\boxed{\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$ will give us the value of the slope at $x = a$, or $(a, f(a))$

and $\boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$ will give us the equation of the slope at any x-value.

This is the DERIVATIVE!

(equation of the slope)

$\boxed{\text{ex1}}$ Find the derivative of $f(x) = 2x^2 + 1$ and then find the slope of the tangent line at $x = \frac{1}{4}$.

$$\lim_{h \to 0} \frac{2(x+h)^2 + 1 - (2x^2+1)}{h} = \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 \cancel{+1} \cancel{-2x^2} \cancel{-1}}{h}$$

$$\lim_{h \to 0} \frac{4xh + 2h^2}{h} = \lim_{h \to 0} 4x + 2h = 4x + 2(0) = \boxed{4x}$$

equation for the slope at any given x-value!

at $x = \frac{1}{4}$ ... $m = 4\left(\frac{1}{4}\right) = \boxed{1}$

B. All About the Derivative

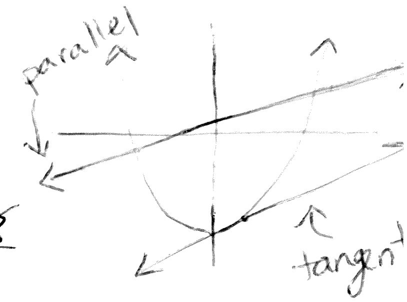
1. The <u>secant's</u> slope is the <u>average</u> rate of change
2. The <u>tangent's</u> slope is the <u>instantaneous</u> rate of change.

3. Notation "Take the derivative" ↘

$f'(x)$	"f prime of x"	...of $f(x)$
$\frac{dy}{dx}$	"d-y-d-x"	...of y with respect to x
$\frac{ds}{dt}$	"d-s-d-t"	...of s with respect to t → physics application
$D_x y$	"d sub x of y"	...of y with respect to x
$\frac{d}{dx} f(x)$	"d-d-x of f(x)"	...of f(x) with respect to x

<u>Differentiation</u> is the process; <u>derivative</u> is the answer

$\boxed{\text{ex2}}$ Find the equation of the tangent line to the curve $y = x^2 - 8$ that is parallel to the line $4x - 2y = 3$. (AP Question)



★ equation for the slope of the tangent:

$$\lim_{h \to 0} \frac{(x+h)^2 - 8 - (x^2-8)}{h} = \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{- 8} \cancel{- x^2} \cancel{+ 8}}{h}$$

$$\lim_{h \to 0} 2x + h = 2x + 0 = 2x$$

★ parallel {same slope} to:

$4x - 2y = 3$

$y = 2x - \frac{3}{2}$

$m = 2$

★ what x-value?

$2x = 2$

$x = 1$

★ coordinate pair {y-value} at $x=1$:

$y = x^2 - 8$

$y = 1^2 - 8 = -7$ $\qquad (1,-7)$

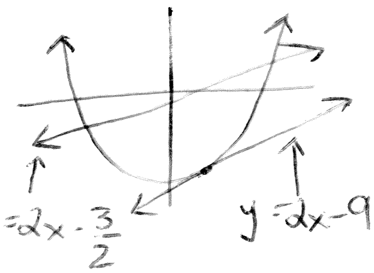
★ equation of the ~~the~~ tangent line ?

point $(1,-7)$ slope $m=2$

$y = mx + b$

$-7 = 2(1) + b$ $\qquad \boxed{y = 2x - 9}$

$-9 = b$



C. Normal Lines

A normal line is another name for a perpendicular line to the tangent.

$\boxed{\text{ex1}}$ Find the ~~equation~~ of the normal line to $f(x) = x^2 + 3x - 2$ at $x = 1$.

slope of the tangent $\Rightarrow$ derivative! $\lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) - 2 - (x^2 + 3x - 2)}{h} =$

$\lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{3x} + 3h \cancel{- 2} - \cancel{x^2} - \cancel{3x} \cancel{+ 2}}{h} = \lim_{h \to 0} 2x + h + 3 = 2x + 0 + 3$

$= 2x + 3$

tangent slope at $x=1$: $m = 2(1) + 3 = 5$

normal slope at $x=1$: $m = -\frac{1}{5}$

point: $f(1) = 1^2 + 3(1) - 2 = 2$ $\qquad (1,2)$

$2 = -\frac{1}{5}(1) + b$

$\frac{11}{5} = b$

$\boxed{y = -\frac{1}{5}x + \frac{11}{5}}$

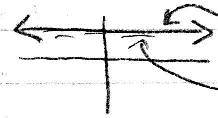
II Derivative Rules (2C)

A. The derivative of a constant

$\boxed{\frac{d}{dx}(c) = 0}$ or $\boxed{\text{If } f(x) = c, \text{ then } f'(x) = 0}$

$\boxed{\text{ex1}}$ $\frac{d}{dx}(4) = 0$

$\boxed{\text{ex2}}$ If $f(x) = 5^k$, where k is a constant, $f'(x) = 0$

Visual: 

constant function

the tangent lines all have a slope of 0

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

≥ proof ≤ If $f(x) = c$, using the definition of the derivative:

$$\lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

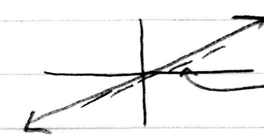
AWAD!

B. The derivative of a linear function

$\boxed{\frac{d}{dx}(cx + b) = c}$ or $\boxed{\text{If } f(x) = cx + b, \ f'(x) = c}$

$\boxed{\text{ex3}}$ If $f(x) = 22x + 5$, $f'(x) = 22$

$\boxed{\text{ex4}}$ If $f(x) = \frac{1}{2}x$, $f'(x) = \frac{1}{2}$

Visual: 

linear function

tangent lines will all have the same slope as the line!

≥ proof ≤ If $f(x) = cx + b$

$$\lim_{h \to 0} \frac{c(x+h) + b - (cx + b)}{h} = \lim_{h \to 0} \frac{\cancel{cx} + ch + \cancel{b} - \cancel{cx} - \cancel{b}}{h}$$

$$= \lim_{h \to 0} \frac{ch}{h} = \lim_{h \to 0} c = c.$$

AWAD!

E. The Derivative of a Product

$$\boxed{\left[f(x)\cdot g(x)\right]' = f(x)\cdot g'(x) + g(x)\cdot f'(x)}$$

$$\boxed{\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)}$$

ex 9 $f(x) = \underbrace{(x^3+5x+7)}_{1^{st}}\underbrace{(x^3+6x+8)}_{2^{nd}}$

$$f'(x) = (x^3+5x+7)(x^3+6x+8)' + (x^3+6x+8)(x^3+5x+7)'$$
$$= \boxed{(x^3+5x+7)(3x^2+6) + (x^3+6x+8)(3x^2+5)}$$

proof: $\left(f(x)\cdot g(x)\right)'$

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$=0$ ninja

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) \overbrace{- f(x+h)g(x) + f(x+h)g(x)}^{=0} - f(x)g(x)}{h}$$

$$\lim_{h\to 0} \frac{f(x+h)\left[g(x+h) - g(x)\right] + g(x)\left[f(x+h) - f(x)\right]}{h}$$

$$\lim_{h\to 0} \underbrace{f(x+h)}_{f(x)}\underbrace{\frac{\left[g(x+h) - g(x)\right]}{h}}_{\text{derivative of } g(x)} + \lim_{h\to 0} \underbrace{g(x)}_{g(x)}\underbrace{\frac{\left[f(x+h) - f(x)\right]}{h}}_{\text{derivative of } f(x)}$$

$$= f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

AWAD!

F. The Derivative of a Quotient

$$\boxed{\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}}$$

$$\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}}$$

ex 10 $f(x) = \frac{x-3}{x^2+1}$

$$f'(x) = \frac{(x^2+1)(x-3)' - (x-3)(x^2+1)'}{(x^2+1)^2} = \frac{(x^2+1)(1) - (x-3)(2x)}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2+6x}{(x^2+1)^2} = \boxed{\frac{-x^2+6x+1}{(x^2+1)^2}}$$

proof: $\frac{f(x)}{g(x)}$

$$\lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{h\,g(x)g(x+h)}$$

=0 Ninja!

$$\lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h\,g(x)g(x+h)}$$

$$\lim_{h\to 0}\frac{\overset{\to g(x)}{g(x)}\overset{\to f'(x)}{\boxed{f(x+h)-f(x)}}}{h\underset{\Downarrow\, g(x)}{g(x)}\underset{\Downarrow\, g(x)}{g(x+h)}} + \lim_{h\to 0}\frac{\overset{\to -f(x)}{-f(x)}\overset{\to g'(x)}{\boxed{g(x+h)-g(x)}}}{h\underset{\Downarrow\, g(x)}{g(x)}\underset{\Downarrow\, g(x)}{g(x+h)}}$$

$$\frac{g(x)\cdot f'(x)}{(g(x))^2} + \frac{-f(x)\cdot g'(x)}{(g(x))^2} = \frac{g(x)\cdot f'(x) - f(x)g'(x)}{[g(x)]^2}$$

AWAD!

G. Extensions

1. Combination

quotient rule

$$\frac{d}{dx}\left(\frac{e^x x^2+1}{3x}\right) = \frac{(3x)(e^x x^2+1)' - (e^x x^2+1)(3x)'}{(3x)^2}$$

Product rule

$$\frac{(3x)(e^x \cdot 2x + x^2 \cdot e^x) - (e^x x^2+1)(3)}{9x^2} = \frac{6e^x x^2 + 3x^3 e^x - 3e^x x^2 - 3}{9x^2}$$

$$= \frac{3e^x x^2 + 3e^x x^3 - 3}{9x^2} = \boxed{\frac{e^x x^2 + e^x x^3 - 1}{3x^2}}$$

2. Extension of the Product Rule

$$\boxed{[f(x)\cdot g(x)\cdot h(x)]' = f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x)}$$

3. Derivatives to Memorize

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

memorize for speed!

4. The Derivative at a Point

$$\frac{d}{dx}\left(e^x(3x^2-x+1)\right)\Big|_{x=1}$$

means evaluate the derivative at $x=1$

this will find the slope of the tangent line at $x=1$.

$$e^x(6x-1) + (3x^2-x+1)e^x \Big|_{x=1}$$

$$e^1(6(1)-1) + (3(1)^2-(1)+1)e^1$$

$$5e + 3e = \boxed{8e}$$

H. Trigonometric Derivatives (2D)

$\frac{d}{dx}(\sin x) = \cos x$ $\qquad$ $\frac{d}{dx}(\csc x) = -\csc x \cot x$

$\frac{d}{dx}(\cos x) = -\sin x$ $\qquad$ $\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\qquad$ $\frac{d}{dx}(\cot x) = -\csc^2 x$

proof: $((\cos(x))'$

SPID

$$\lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \to 0} \frac{\cos x \cosh - \sin x \sinh - \cos x}{h} =$$

$$\lim_{h \to 0} \frac{\cos x \cosh - \cos x}{h} - \frac{\sin x \sinh}{h} = \lim_{h \to 0} \frac{-\cos x(-\cosh + 1)}{h} - \frac{\sin x \sin h}{h}$$

$$\lim_{h \to 0} -\cos x \underbrace{\frac{(-\cosh + 1)}{h}}_{\to 0} - \sin x \underbrace{\frac{\sinh}{h}}_{\to 1}$$

$$\lim_{h \to 0} \frac{1 - \cosh}{h} = 0$$

$$\lim_{h \to 0} \frac{\sinh}{h} = 1$$

$= -\sin x$

AWAP!

I. The Derivative of a Composition Function (2E)

If $F(x) = f(g(x))$ ← composition!
then
$$\boxed{F'(x) = f'(g(x)) \cdot g'(x)}$$

or

If $y = f(u)$ and $u = g(x)$ ← composition!
then
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$\boxed{\text{ex1}}$ $f(x) = (x^3 + 2x + 5)^{1/2}$

outside: $x^{1/2}$
inside: $x^3 + 2x + 5$

$$f'(x) = \boxed{\frac{1}{2}(x^3+2x+5)^{-1/2} \cdot (3x^2+2)}$$
$$= \boxed{\frac{3x^2+2}{2(x^3+2x+5)^{1/2}}}$$

$\boxed{\text{ex2}}$ $f(x) = e^{8x^2+1}$

outside: e^x
inside: $8x^2+1$

$$f'(x) = e^{8x^2+1} \cdot (16x)$$
$$= \boxed{16x\, e^{8x^2+1}}$$

$\boxed{\text{ex3}}$ $f(x) = \cos(\ln(x^2+1))$

oh dang! Triple Chain!
$f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$

$$f'(x) = -\sin(\ln(x^2+1)) \cdot \frac{1}{x^2+1} \cdot 2x$$

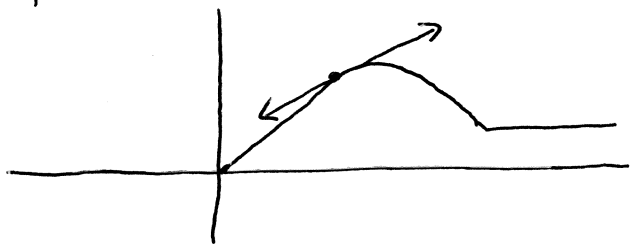
$$= \boxed{\frac{-2x \sin(\ln(x^2+1))}{x^2+1}}$$

III. Particle Motion

A "particle" is just indicating a refrence point on a moving object. 

sometimes $x(t)$

Given the graph of the position of a particle, $s(t)$.



The slope of the tangent line at a given point is the rate of change in the position.

A. Instantaneous Velocity - the rate of change in position at one point in time.

so $s'(t) = v(t)$

the derivative of the position function is the velocity function

Note: ★ speed $\neq$ velocity
velocity has direction, so if its negative, that means it's moving backwards.
$|\text{velocity}| = \text{speed}$

★ instantaneous velocity $\neq$ average velocity

$$\text{average velocity} = \frac{\Delta \text{position}}{\Delta \text{time}} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

instantaneous velocity = derivative!

B. Acceleration - the rate of change in the velocity at one point in time.

$$s''(t) = v'(t) = a(t)$$

C. Jerk - rate of change in the acceleration

$$s'''(t) = v''(t) = a'(t) = j(t)$$

D. Describing a Particle's Motion

1. Right/Forward: $v(t) > 0$
2. Left/Backward: $v(t) < 0$
3. Stopped/At rest: $v(t) = 0$
4. Speeding Up:
   a. Speeding up moving forward:
      $v(t) > 0$ and $a(t) > 0$ $\quad +,+$
   b. Speeding up moving backward:
      $v(t) < 0$ and $a(t) < 0$ $\quad -,-$

   same sign
5. Slowing Down:
   a. slowing down moving forward:
      $v(t) > 0$ and $a(t) < 0$ $\quad +,-$
   b. Slowing down moving backward
      $v(t) < 0$ and $a(t) > 0$ $\quad -,+$

   opposite sign

$\boxed{ex\ 1}$ A particle is moving along a horizontal line according to the position function $s(t) = 4t^3 - t^2$ given in meters at t seconds. What is the velocity of the particle after 5 seconds?

$$s(t) = 4t^3 - t^2$$

$$s'(t) = v(t) = 12t^2 - 2t$$

$$v(5) = 12(5)^2 - 2(5) = \boxed{290 \text{ m/s}}$$

$\boxed{\text{ex2}}$ A particle moves on a horizontal line according to $s(t) = \frac{1}{3}t^3 - t$. When is the particle speeding up? Explain your reasoning.

Approach: Find $v(t)$ and $a(t)$ and their critical points {set $=0$}. Use a chart to determine on what interval each one is either $+$, $-$ or 0.

$s(t) = \frac{1}{3}t^3 - t$

$s'(t) = v(t) = t^2 - 1 \longrightarrow$ c.p. $t^2 - 1 = 0$, $t^2 = 1$, $t = \pm 1$

$s''(t) = a(t) = 2t \rightarrow$ c.p. $2t = 0$, $t = 0$

time	$\frac{1}{3}t^3 - t$ $s(t)$	t^2-1 $v(t)$	$2t$ $a(t)$	conclusion
$(-\infty, -1)$		$+$	$-$	moving forward, slowing down
-1	$\frac{2}{3}$	0	$-$	at rest
$(-1, 0)$		$-$	$-$	moving backward, speeding up
0	0	$-$	0	moving backward, constant speed
$(0, 1)$		$-$	$+$	moving backward, slowing down
1	$-\frac{2}{3}$	0	$+$	at rest
$(1, \infty)$		$+$	$+$	moving forward, speeding up

not always necessary!

Note: For the AP test, a chart is <u>not</u> an explanation

Answer: The particle is speeding up on the interval $(-1, 0) \cup (1, \infty)$ because the velocity & acceleration have the same sign.

VIII. Physics Applications (2I)

A. Position: $s(t)$

B. Velocity: $v(t) = s'(t)$

C. Acceleration: $a(t) = v'(t) = s''(t)$

D. Height of a Projectile:

sometimes x_0

$$s(t) = -\frac{1}{2}at^2 + v_0 t + s_0$$

height at any given time, t

acceleration due to gravity

initial velocity

initial height

$$v(t) = -at + v_0$$

$$a(t) = -a$$

using feet per second^2, $a = 32\ ft/s^2$

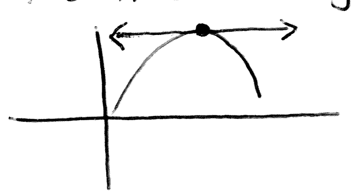
$$\boxed{s(t) = -16t^2 + v_0 t + s_0}$$

E. 3 things we need to know how to do:

1. Speed $= |v(t)|$ {no direction}

2. Average velocity: $\dfrac{\Delta \text{position}}{\Delta \text{time}} = \dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$

3. Maximum height:



maximums occur when the slope of the tangent line is 0, aka $v(t) = 0$.

ex1 A dynamite blast propels a rock straight up with a launch velocity of 160 ft/s.

a. How high did the rock go?

b. What is the velocity and speed when the rock is 256 ft up?

c. What is the acceleration of the rock at any time?

d. When does the rock hit the ground?

e. What is the average velocity from $t=0$ to $t=2$?

$$S(t) = -16t^2 + 160t$$

a. max: $V(t)=0$

$$V(t) = -32t + 160 = 0$$
$$-32t = -160$$
$$t = 5$$

$$S(5) = -16(5)^2 + 160(5)$$
$$= \boxed{400 \text{ ft}}$$

b. velocity @ 256 ft: find t first!

$$S(t) = -16t^2 + 160t = 256$$
$$-16t^2 + 160t - 256 = 0$$
$$-16(t^2 - 10t + 16) = 0$$
$$-16(t-8)(t-2) = 0$$
$$t=8 \quad t=2$$

$$V(8) = -32(8) + 160 = \boxed{-96 \text{ ft/s}}$$
{down}

$$V(2) = -32(2) + 160 = \boxed{96 \text{ ft/s}}$$
{up}

$$\boxed{\text{Speed} = 96 \text{ ft/s}}$$

c. $a(t) = v'(t)$

$$a(t) = -32$$
$$\boxed{-32 \text{ ft/s}^2}$$

d. $S(t)=0$ means ground!

$$-16t^2 + 160t = 0$$
$$-16t(t-10) = 0$$
$$t=0 \quad t=10$$
$$\boxed{10 \text{ sec}}$$

e. ave. vel. $\frac{\Delta \text{pos}}{\Delta \text{time}}$

$$\frac{S(2)-S(0)}{2-0} = \frac{256-0}{2-0} = \boxed{128 \text{ ft/s}}$$

IV. Implicit Differentiation (2J)

A. When you cannot seperate/solve for y, we need to differentiate both sides of the equation with respect to x {x is the variable}. Keep in mind that y is a function, like $f(x)$.

ex1 For $x^2+y^2=7$, what is y'? {or $\frac{dy}{dx}$}

→ we can solve for y, or differentiate now!

$$2x+2y\cdot y'=0$$

it's the chain rule:
$\left((f(x))^2\right)' = 2f(x)\cdot f'(x)$
$(y^2)' = 2y\cdot y'$

$$2y\cdot y' = -2x$$

$$y' = \frac{-2x}{2y}$$

$$\boxed{y' = -\frac{x}{y}}$$

ex2 Find the tangent line to $x^3+y^3=6xy$ at the point $(3,3)$.

Use the product rule!

$$3x^2+3y^2\cdot y' = 6x\cdot y' + 6y$$

evaluate the slope @ (3,3) →

$$3(3)^2+3(3)^2\cdot y' = 6(3)\cdot y' + 6(3)$$

$$27+27y' = 18y'+18$$

$$9y' = -9$$

$$y' = -1$$

$$3 = -1(3)+b$$

$$6 = b$$

$$\boxed{y = -x+6}$$

ex3 Find $\frac{d^2y}{dx^2}$ if $y^2 + e^y = x$

2nd derivative, can't solve for y ∴ implicit differentiation

$$2y \cdot y' + e^y \cdot y' = 1$$

$$y'(2y + e^y) = 1$$

$$y' = \frac{1}{2y + e^y} = (2y + e^y)^{-1}$$

replace with y'

$$y'' = -(2y + e^y)^{-2} \cdot (2y' + e^y \cdot y')$$

$$y'' = -(2y + e^y)^{-2} \cdot \left[2(2y + e^y)^{-1} + e^y(2y + e^y)^{-1}\right]$$

$$\boxed{y'' = -(2y + e^y)^{-3}(2 + e^y)}$$

V. Logarithmic Differentiation (2#)

when the "respected" variable is in the exponent, what will bring them down ?!

Logarithims!

we need logarithmic and implicit differentiation.

A. Using Logs to Differentiate

$\boxed{\text{ex 1}}$ Find $\frac{dy}{dx}$ if $y = x^x$

$\ln y = \ln x^x$ ← ln both sides

$\ln y = x \ln x$ ← product rule

implicit differentiation → $\frac{1}{y} \cdot y' = x \cdot \frac{1}{x} + \ln x \cdot (1)$

$\frac{1}{y} \cdot y' = 1 + \ln x$

$y' = (1 + \ln x) y$ ← always replace y if possible

$\boxed{y' = (1 + \ln x) x^x}$

$\boxed{\text{ex2}}$ Find $g'(x)$ if $g(x) = 2^{5x} \cdot 3^{4x^2}$

$y = 2^{5x} \cdot 3^{4x^2}$

$\ln y = \ln (2^{5x} \cdot 3^{4x^2})$

$\ln y = \ln (2^{5x}) + \ln (3^{4x^2})$

$\ln y = 5x \ln 2 + 4x^2 \ln 3$

$\frac{1}{y} \cdot y' = 5 \ln 2 + 8x \ln 3$

$y' = (5 \ln 2 + 8x \ln 3) y$

$\boxed{y' = (5 \ln 2 + 8x \ln 3)(2^{5x} \cdot 3^{4x^2})}$

B. Rule: Memorize & use it

If $f(x) = a^x$, then $f'(x) = \ln a \cdot a^x$

{proof}

$$y = a^x$$
$$\ln y = \ln a^x$$
$$\ln y = x \ln a$$
$$\frac{1}{y} \cdot y' = \ln a$$
$$y' = \ln a \cdot y$$
$$y' = \ln a \cdot a^x$$

AWAD!

$\boxed{ex\ 3}$ $f(x) = 3^x$

$$\boxed{f'(x) = \ln 3 \cdot 3^x}$$

Rule:

If $f(x) = \log_a x$ then $f'(x) = \frac{1}{x \ln a}$

$\boxed{ex\ 4}$ $f(x) = \log_5 3x$

$$f'(x) = \frac{1}{3x \ln 5} \cdot 3 = \boxed{\frac{1}{x \ln 5}}$$

VI. Derivatives of Inverse Trig Functions (2K)

Rules:

$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\csc^{-1}(x)) = \frac{-1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}(\cos^{-1}(x)) = \frac{-1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$ $\frac{d}{dx}(\cot^{-1}(x)) = \frac{-1}{1+x^2}$

Note: $\sin^{-1}(x) \neq \frac{1}{\sin x}$ but $\sin^{-1}(x) = \arcsin(x)$

to find the inverse sine function.

$y = \sin x$

$x = \sin y$

$\sin^{-1}(x) = y$

★ proof: $y = \sin^{-1} x$

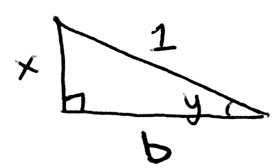
by the definition of inverse, this means that

$x = \sin y$

and by implicit differentiation:

$1 = \cos y \cdot y'$

$y' = \frac{1}{\cos y}$

Because $x = \sin y$, 

so $\cos y = \frac{b}{1}$ and

by the pythagorean theorem $b^2 + x^2 = 1^2$ so $\cos y = \sqrt{1-x}$

$b = \sqrt{1-x^2}$

By substitution, $y' = \frac{1}{\sqrt{1-x^2}}$. AWAD!

$\boxed{\text{ex 1}}$ $\frac{d}{dx}\left(\sin^{-1}(4x^3+1)\right)$

Approach: chain rule & inverse trig derivs

$$(\sin^{-1}(x))' = \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1}{\sqrt{1-(4x^3+1)^2}} \cdot 12x^2 = \boxed{\frac{12x^2}{\sqrt{1-(4x^3+1)^2}}}$$

$\boxed{\text{ex 2}}$ $\left[\tan^{-1}(4\pi x)\right]' = \boxed{\frac{4\pi}{1+(4\pi x)^2}}$

VII. L'Hopital's Rule (2F)

Rule:

If $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]$ has an indeterminate form

of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \lim_{x\to a}\left[\frac{f'(x)}{g'(x)}\right]$$

this only works with limits!

don't confuse this with the quotient rule!

*proof for $\frac{0}{0}$ case

Assume that f' and g' are continuous and $f(a)=g(a)=0$

$$\lim_{x\to a}\left[\frac{f'(x)}{g'(x)}\right] = \frac{f'(a)}{g'(a)} = \frac{\lim\limits_{x\to a}\frac{f(x)-f(a)}{\cancel{x-a}}}{\lim\limits_{x\to a}\frac{g(x)-g(a)}{\cancel{x-a}}}$$

def. of the derivative

b/c f' and g' are continuous

$$= \frac{\lim\limits_{x\to a} f(x)-\cancel{f(a)}}{\lim\limits_{x\to a} g(x)-\cancel{g(a)}} = \lim_{x\to a}\frac{f(x)}{g(x)}$$

AWAD!

$\boxed{\text{ex1}}$ $\lim_{x\to 2}\frac{x^3-8}{x-2} = \frac{0}{0}$ ∴ indeterminate form

① Algebra
② L'Hopital's Rule!

$$\lim_{x\to 2}\frac{3x^2}{1} = 3(2)^2 = \boxed{12}$$

$\boxed{\text{ex2}}$ $\lim_{x\to 0} x\cdot\ln x = 0\cdot -\infty$ ∴ indeterminate, but not a ratio

$$\lim_{x\to 0}\frac{\ln x}{x^{-1}} \overset{L'H}{=} \lim_{x\to 0}\frac{\frac{1}{x}}{-x^{-2}} = \lim_{x\to 0}\frac{1}{x}\cdot -x^2 = \lim_{x\to 0} -x = \boxed{0}$$

$\boxed{ex3}$ $\lim_{x \to \infty} \frac{2e^x}{1+x^3} = \frac{\infty}{\infty}$ L'H !

$$\lim_{x \to \infty} \frac{2e^x}{3x^2} \overset{L'H}{=} \lim_{x \to \infty} \frac{2e^x}{6x} \overset{L'H}{=} \lim_{x \to \infty} \frac{2e^x}{6} = \frac{\infty}{6} = \boxed{\infty}$$

$\boxed{DNE}$

IX. Differentiability (2L)

Prior Knowledge: Continuity

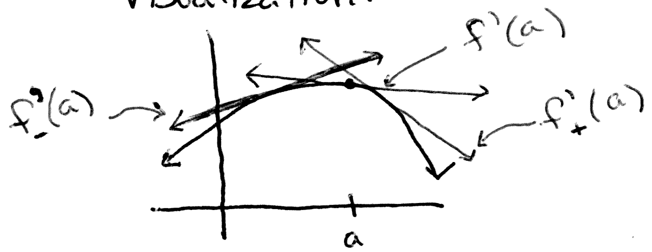
If $f(a)$ exists, $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = f(a)$

then $f(x)$ is continuous.

A. Differentiability

the ability to take the derivative

Visualization:



Definition:

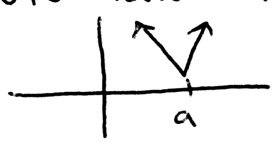
A function $f(x)$ is differentiable at a if $f'(a)$ exists, which means $f'_-(a) = f'_+(a)$

derivative from the left

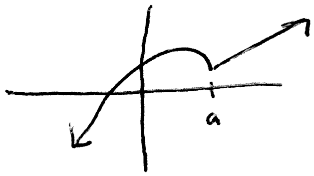
from the right

B. Non-Differentiable Functions

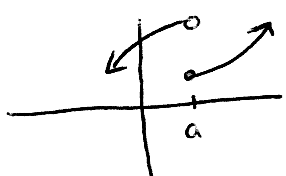
1. Absolute Value Functions



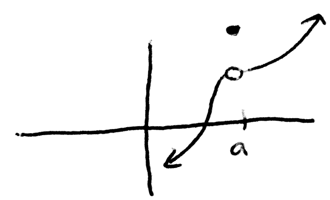
2. Piecewise functions with...



a cusp/corner

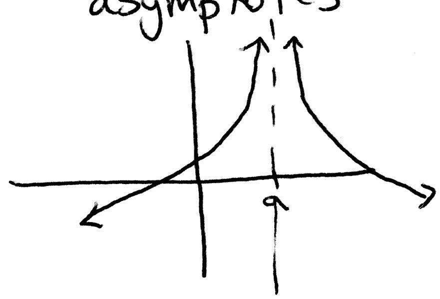


a jump discontinuity

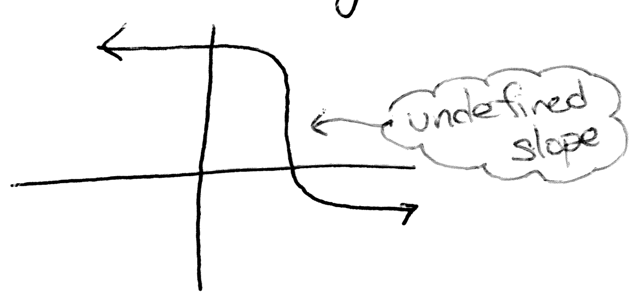


a removable discontinuity

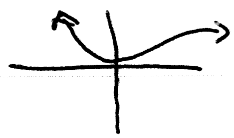
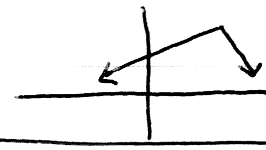
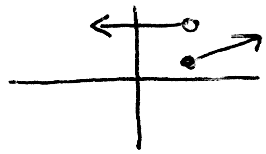
3. Rational Functions with holes or asymptotes



4. Functions with a Vertical Tangent Line



C. The Connection Between Differentiability & Continuity

Differentiable & Continuous 	Not Differentiable & continuous 
Not Differentiable, Not continuous 	Differentiable & Not Continuous ?! not possible!

Conclusion: If $f(x)$ is differentiable, then it is continuous.

{ the converse is NOT true }

D. Showing Differentiability Analytically ← Using Math

Goal: Show that $f'_-(a) = f'_+(a)$

ex 1 Where is $f(x) = |x+2|$ differentiable?

Approach: ① write as a piecewise ② take the derivative

$$f(x) = \begin{cases} x+2 & x \geq -2 \\ -x-2 & x < -2 \end{cases}$$ ← concerned @ $x=-2$

$$f'(x) = \begin{cases} 1 & x \geq -2 \\ -1 & x < -2 \end{cases}$$

$f'_-(-2) = -1$

$f'_+(-2) = 1$

since $f'_-(-2) \neq f'_+(-2)$ the function is not differentiable at $x=-2$.

differentiable on the interval $\boxed{(-\infty,-2) \cup (-2,\infty)}$

$\boxed{\text{ex2}}$ where is $f(x) = \dfrac{2x}{x^2-1}$ differentiable?

Approach: ① check the denom. for continuity
② take the derivative

$f(x) = \dfrac{2x}{(x+1)(x-1)}$ not continuous at $x = \pm 1$

$\therefore$ not differentiable at $x = \pm 1$

$f'(x) = \dfrac{(x^2-1)(2) - (2x)(2x)}{(x^2-1)^2}$ ← same restricted domain!

$\boxed{(-\infty,-1)\cup(-1,1)\cup(1,\infty)}$

$\boxed{\text{ex3}}$ Find the values of a and b such that $f(x)$ is differentiable everywhere.

$f(x) = \begin{cases} 3-x & x<1 \\ ax^2+bx & x \geq 1 \end{cases}$ ← concerned at $x=1$

Approach: ① check for continuity ② check for differentiability

① Continuity: $f(1) = a(1)^2 + b(1) = a+b$

$\lim\limits_{x\to 1^-} f(x) = 3-1 = 2$ $\therefore a+b=2$

$\lim\limits_{x\to 1^+} f(x) = a(1)^2 + b(1) = a+b$

system!

② Differentiability: $f'(x) = \begin{cases} -1 & x<1 \\ 2ax+b & x \geq 1 \end{cases}$

$f'_-(1) = -1$ $\therefore 2a+b = -1$

$f'_+(1) = 2a(1) + b = 2a+b$

$$\begin{array}{r} 2a+b=-1 \\ -\ a-b=-2 \\ \hline \boxed{a=-3} \end{array}$$

$-3+b=2$

$\boxed{b=5}$

X. Solving, Derivatives & Limits on the TI-89

#SorryI'mnotsorry

A. Limits: Regular & Intederminate Form

ex1 $\lim_{h \to 0} \frac{\sqrt{1+h}-1}{h}$

type in the function

F3 limit f(x), x, 0) enter $= \boxed{\frac{1}{2}}$

comma

what the variable approaches

B. Limits: To Infinity

ex2 $\lim_{x \to -\infty} \left(\frac{6}{\sqrt[3]{x}} + \frac{1}{\sqrt[5]{x}} \right)$

$x^{1/5}$

F3 limit f(x), x, − ◆catalog) enter $= \boxed{0}$

C. Derivatives: Regular

ex3 $\frac{d}{dx}\left(e^{2x+1} \tan x \right)$

what variable do you respect?

F3 differentiate f(x), x) enter $= \boxed{\frac{(2\sin x \cos x + 1)e^{2x+1}}{(\cos(x))^2}}$

D. Derivatives: Implicit

ex4 Find $\frac{dy}{dx}$ if $1+x=\sin(xy^2)$

must use the multiplication symbol here

F3 impDif f(x), x, y) enter

with respect to

derivative of...

E. Evaluating

$\boxed{\text{ex5}}$ Find the slope of the tangent line to

$f(x) = (x^3+3)\cdot 2^{-7x}$ at $x=2$.

"such that"

(F3) differentiate $f(x)$, x) | $x=2$ $\boxed{\text{enter}}$

$$\boxed{\frac{-(77\ln 2-12)}{16,384}}$$

$\boxed{-.00252}$

F. Solving Algebraic/Other Equations

$\boxed{\text{ex6}}$ Solve $4x+e^x=5$ for x.

(F2) solve $f(x)$, x) $\boxed{\text{enter}}$ $= \boxed{.731}$