Unit 1 Limits & Continuity

8/16

What is calculus?

A. Can we reach infinity?

B. Finding areas between curves

C. Motion & changes in motion

I. Limits (1A)

A. Dictionary Defn: (n) the point, edge or line beyond which something can't or may not proceed.

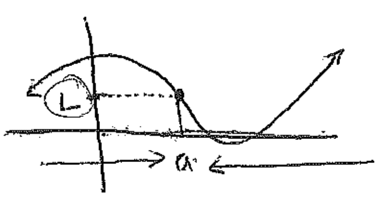
B. Calculus Defn:

$\lim_{x \to a} f(x) = L$ mean: "the limit of f(x) as x approaches a is L"

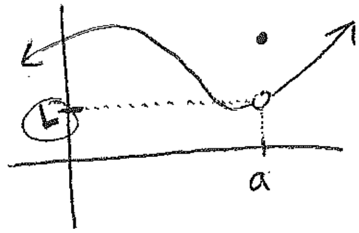
closer & closer... but not a

matter: As the x-values approach a, f(x) or your y-values approach L.

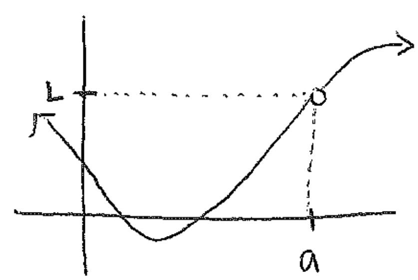
C. Visual



normal

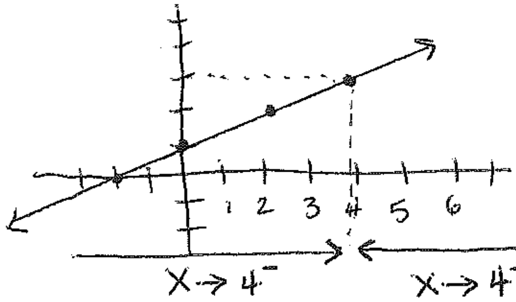


piecewise type function where the actual value of f(x) at a is different than what f(x) approaches near a.



"hole" in the function and the value of f(x) isn't defined at a but the limit still exists.

real ex $\lim_{x \to 4} \frac{1}{2}x + 1$



* the x-values can approach from the left:

$$\lim_{x \to 4^-} f(x) = 3$$

* or the right

$$\lim_{x \to 4^+} f(x) = 3$$

* both sides approach 3.

$$\therefore \lim_{x \to 4} \frac{1}{2}x + 1 = \boxed{3}$$

RULE:

If $\lim_{x \to a^-} f(x) = L$

and $\lim_{x \to a^+} f(x) = L$

then $\lim_{x \to a} f(x) = L$

if not... $\therefore$ $\lim_{x \to a} f(x) = \text{DNE}$

D. Estimating Limits Using Tables

Find the Value of $\lim_{x \to 1} \frac{x-1}{x^2-1}$

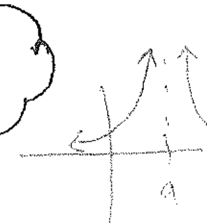
notice 1 is not in the domain... so you can't just plug it in

if	x	f(x)
$x \to 1^-$	.7	.58825
	.8	.5556
	.9	.52632
	1	undefined
$x \to 1^+$	1.1	.47619
	1.2	.45455
	1.3	.43478

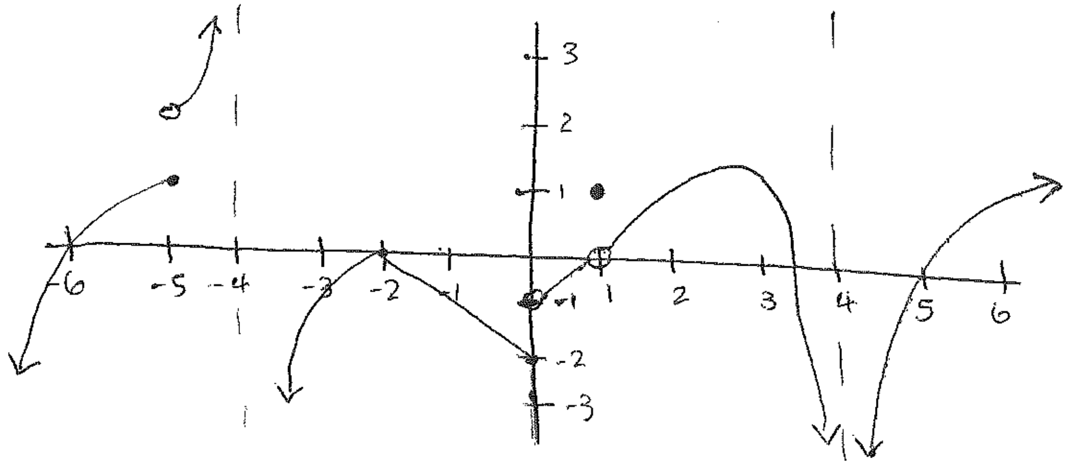
← closer & closer to .5

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = \boxed{.5}$$

Note: If $\lim_{x \to a} f(x) = \infty$ ... then $\lim_{x \to a} f(x) = \text{DNE}$



E. One Sided Limits



① what is $\lim_{x \to -5} f(x) = ?$

from the left: $\lim_{x \to -5^-} f(x) = 1$ from the right: $\lim_{x \to -5^+} f(x) = 2$

not the same ... So $\lim_{x \to -5} f(x) = \boxed{DNE}$ even though $f(-5) = 1$

② what is $\lim_{x \to 4} f(x) = ?$

left: $\lim_{x \to 4^-} f(x) = -\infty$ right: $\lim_{x \to 4^+} f(x) = -\infty$

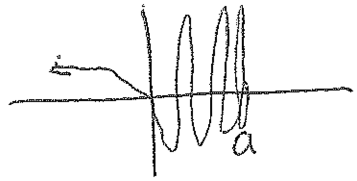
~~not the same~~ but $-\infty$ is not a number... so $\lim_{x \to 4} f(x) = \boxed{DN}$

③ what is $\lim_{x \to 0^+} f(x) = \boxed{-1}$

Remember: ★ if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ then $\lim_{x \to a} f(x) = DNE$

★ if $\lim_{x \to a} f(x) = \infty$ then $\lim_{x \to a} f(x) = DNE$

★ if



then $\lim_{x \to a^-} f(x) = DNE$

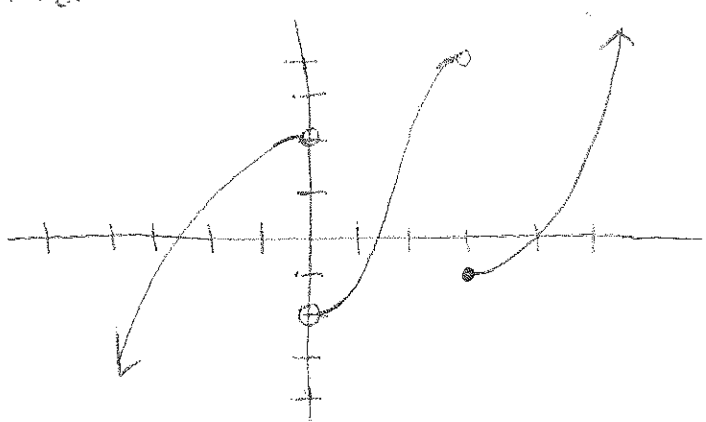
because f(x) is oscilating and not approaching a specific y-value

F Sketch a Graph Given the Limits

$\boxed{\text{ex1}}$ Sketch a function f(x) such that $\lim_{x \to 3^-} f(x) = 4$

$f(3) = -1$, $\lim_{x \to 0^-} f(x) = 2$, $\lim_{x \to 0^+} f(x) = -2$

$\lim_{x \to a} f(x) = f(a)$ if $x \neq 0, 3$ and the range is $\mathbb{R}$.



8/21

II Calculating Limits Using Limit Theorems (1D)

* A picture is <u>not</u> proof.
* Inductive reasoning is <u>not</u> proof {tables}
  → these are estimations of the limit
* we need theorems to prove :)

A. Limit Theorems

"plug in rules"

1. $\lim_{x \to a} x = a$
2. $\lim_{x \to a} x^n = a^n$ {n is a positive integer}
3. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ {n is a positive integer, and if n is even then $a > 0$}
4. $\lim_{x \to a} c = c$ {where c is a constant}
5. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
6. $\lim_{x \to a} [cf(x)] = c \cdot \lim_{x \to a} f(x)$
7. $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
8. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ {$\lim_{x \to a} g(x) \neq 0$}
9. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$ {n is a postive integer}
10. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ {n is a positive integer and if n is even $\lim_{x \to a} f(x) > 0$}

$\boxed{\text{ex1}}$ Justify the limit is the indicated number.

$$\lim_{x \to 1} \frac{x(x+3)}{x+1} = 2$$

$$\lim_{x\to 1} \frac{x(x+3)}{x+1} \overset{\#8}{=} \frac{\lim_{x\to 1} x(x+3)}{\lim_{x\to 1} (x+1)} \overset{\#7}{=} \frac{\left[\lim_{x\to 1} x\right] \cdot \left[\lim_{x\to 1} (x+3)\right]}{\lim_{x\to 1} (x+1)} =$$

$$\overset{\#5}{=} \frac{\lim_{x\to 1} x \cdot \left[\lim_{x\to 1} x + \lim_{x\to 1} 3\right]}{\lim_{x\to 1} x + \lim_{x\to 1} 1} = \frac{\overset{\#1}{1} \cdot [\overset{\#1}{1} + \overset{\#4}{3}]}{\underset{\#1}{1} + \underset{\#4}{1}} = \frac{4}{2} = 2$$

B. Justify Vs. Evaluate

1. Justify the limit:

#5

#6   #2 #1 #4

$$\lim_{x\to6}\frac{x^2-2x+1}{x} = \frac{\lim_{x\to6}x^2 - 2\lim_{x\to6}x + \lim_{x\to6}1}{\lim_{x\to6}x} = \frac{(6)^2-2(6)+1}{6}$$

#8   #1

$$= \frac{25}{6}$$

b/c limit theorems allow this!

2. Evaluate the limit

$$\lim_{x\to6}\frac{x^2-2x+1}{x} = \frac{6^2-2(6)+1}{6} = \boxed{\frac{25}{6}}$$

Note: $\lim_{x\to6} f(x) \neq f(6)$ two different ideas...

a. what if no limit theorems work?
→ use graph/table/calculator

b. if you can use limit theorems...
→ "plug in" and simplify

<u>HOWEVER</u>... you may get stuck!

if $\lim_{x\to0} f(x) = \frac{0}{\#} = 0$ good.

if $\lim_{x\to0} f(x) = \frac{\#}{0} = \infty$ $\therefore$ DNE   good.
approaching!

if $\lim_{x\to0} f(x) = \frac{0}{0}$ = ??? ... it's <u>not</u> DNE or 0 ...
(approaching / approaching)

IDK ↑ for now :)

C. Evaluating Limits of Indeterminate Form (IF)

If you use the Limit Theorems to "plug in" and get:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0^0,\ 0\cdot\infty,\ 1^\infty,\ \infty^0,\ \infty\pm\infty$$

this is called <u>indeterminate form</u>, Solve by:

* using algebra rules to simplify / change the format
  → factor / cancel / rationalize / trig IDs / etc

* try limit theorems again

* repeat as necessary

ex 1 $\lim_{x \to 0} \frac{\sqrt{x+4}-2}{x}$ can't "plug in" w/ L.T. $\frac{0}{0}$ ∴ indeterminate form

$$\lim_{x \to 0} \frac{\sqrt{x+4}-2}{x} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \lim_{x \to 0} \frac{x+4-4}{x\sqrt{x+4}+2x} =$$

$$\lim_{x \to 0} \frac{\cancel{x}}{\cancel{x}\sqrt{x+4}+2\cancel{x}} = \lim_{x \to 0} \frac{1}{\sqrt{x+4}+2} = \frac{1}{2+2} = \boxed{\frac{1}{4}}$$

* or use your calculator

f3 → limit → limit (f(x), x, a)

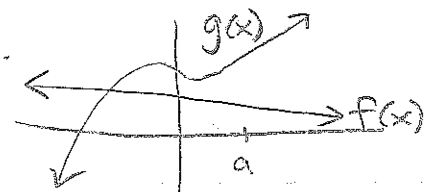
limit $\left(\frac{(\sqrt{x+4}-2)}{x}, x, 0 \right)$

III Additional Theorems for Finding Limits

A. If $f(x) \leq g(x)$ when x is near a {except possibly at a} and the limits of $f(x)$ and $g(x)$ both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

Visual:



near a, $f(x) < g(x)$ ∴ $\lim_{x \to a} f(x) < \lim_{x \to a} g(x)$

aka pinch or sandwitch theorem

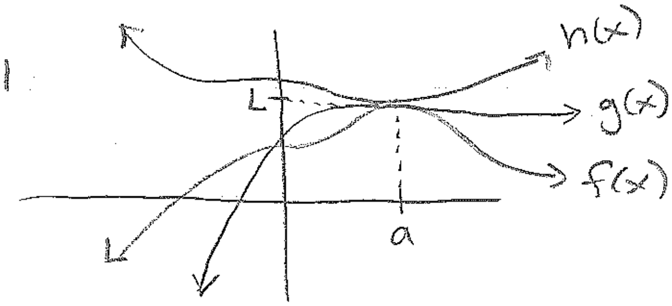
B. The Squeeze Theorem

if $f(x) \leq g(x) \leq h(x)$ when x is near a {except possibly at a} and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then $\lim_{x \to a} g(x) = L$

Visual



near a, $f(x) < g(x) < h(x)$

and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$

then $\lim_{x \to a} g(x) = L$

Useful when you know the limits of $f(x)$ & $h($ but not $g(x)$

C. Trig Limit Theorems

so you can "plug in" for sine & cosine too

1. $\lim_{x \to a} \sin(x) = \sin(a)$

2. $\lim_{x \to a} \cos(x) = \cos(a)$

3. $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ ← from the squeeze theorem

4. $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$

⇒ Proof of #4...

$$\lim_{x \to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x} = \lim_{x \to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x \to 0} \frac{\sin^2 x}{x(1+\cos x)}$$

#3!

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x} = 1 \cdot \frac{0}{2} = 1 \cdot 0 = 0 \quad \checkmark \text{AWAD.}$$

if it was $\frac{\sin(4x)}{4x}$ that's #3...

ex1 $\lim_{x \to 0} \frac{\sin(4x)}{x}$

$$\lim_{x \to 0} \frac{\sin(4x)}{x} \cdot \frac{4}{4} = \lim_{x \to 0} \frac{4\sin(4x)}{4x} = \lim_{x \to 0} 4 \cdot 1 = \boxed{4}$$

all of indeterminate "IDK" form

ex2 $\lim_{x \to 0} \frac{\sin^2 5x}{x^2} = \lim_{x \to 0} \left(\frac{\sin(5x)}{x}\right)^2 = \lim_{x \to 0} \left(\frac{5\sin(5x)}{5x}\right)^2 =$

$$\lim_{x \to 0} (5 \cdot 1)^2 = \lim_{x \to 0} 25 = \boxed{25}$$

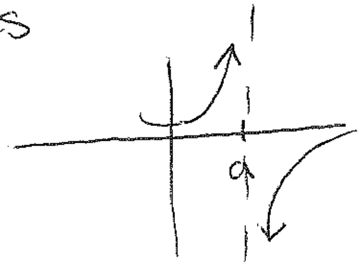
... otherwise just "plug in"

ex3 $\lim_{x \to 0} \frac{1-\cos\sqrt{x}}{\sqrt{x}} = 0$ by #4

IV Limits Involving Infinity & Asymptotes (1F)

A. Vertical Asymptotes

$$\lim_{x \to a} f(x) = \pm\infty$$



Determined by domain

$\boxed{\text{ex1}}$ $f(x) = \dfrac{x}{x-3}$, find the Vertical Asymptote

$x - 3 = 0$

$x = 3$ because $\displaystyle\lim_{x \to 3} \frac{x}{x-3} = \frac{3}{\underset{\text{close to}}{0}} = \infty$

$\dfrac{1}{\underset{\text{close to}}{0}} = \dfrac{1}{\text{really small}} = \infty$

$\dfrac{1}{\infty} = \dfrac{1}{\text{really big}} = 0$

B. Horizontal Asymptotes

$$\lim_{x \to \pm\infty} f(x) = a$$

$\boxed{\text{ex2}}$ $f(x) = \dfrac{x}{x-3}$, find the Horizontal Asymptotes

$\displaystyle\lim_{x \to \pm\infty} \frac{x}{x-3} = \frac{\infty}{\infty - 3} = \frac{\infty}{\infty}$ ... indeterminate form

$\displaystyle\lim_{x \to \pm\infty} \frac{\cancel{x}}{\cancel{x}(1 - \frac{3}{x})} = \lim_{x \to \pm\infty} \frac{1}{1 - \frac{3}{x}} = \frac{1}{1-0} = 1$

using either $\pm\infty$

$\boxed{\therefore\ y = 1 \text{ is the H.A.}}$

ex3 $f(x)=\frac{x+4}{x^2-9}$, Find the H.A.

$$\lim_{x\to\pm\infty}\frac{x+4}{x^2-9}=\lim_{x\to\pm\infty}\frac{\cancel{x^2}\left(\frac{1}{x}+\frac{4}{x^2}\right)}{\cancel{x^2}\left(1-\frac{9}{x^2}\right)}=\frac{0+0}{1+0}=0 \qquad \boxed{\therefore y=0}$$

indeterminate form

ex4 $f(x)=\frac{\sqrt{x^2+2}}{4x-6}$, find H.A.

highest power is x^1 b/c $\sqrt{x^2}=x$

$$\lim_{x\to\pm\infty}\frac{\sqrt{x^2+2}}{4x-6}=\lim_{x\to\pm\infty}\frac{\sqrt{x^2\left(1+\frac{2}{x^2}\right)}}{x\left(4-\frac{6}{x}\right)}=\lim_{x\to\pm\infty}\frac{\cancel{x}\sqrt{1+\frac{2}{x^2}}}{\cancel{x}\left(4-\frac{6}{x}\right)}=$$

$$\frac{\sqrt{1+0}}{4-0}=\frac{1}{4}$$

$$\boxed{\therefore y=\frac{1}{4}}$$

ex5 $\lim_{x\to7^+}\frac{2x}{x-7}=\frac{2\cdot7}{0}=+\infty$

close to (0) but its positive b/c $7.1-7=+.1$

x approaches 7 from the right; bigger than 7

$\lim_{x\to7^-}\frac{2x}{x-7}=\frac{2\cdot7}{0}=-\infty$

close to (0) negative though

$$\lim_{x \to -\infty} e^x = 0$$

$$\lim_{x \to \infty} e^x = \infty$$

If $r > 0$ is a rational #

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

~~If r>0 is~~ and

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

$$\lim_{x \to 0^-} e^{1/x}$$

replace $t = \frac{1}{x}$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$\therefore$

$$\lim_{x \to -\infty} e^t = 0$$

V One-Sided-Limits (1B)

A. Definition of a Limit

If $\lim\limits_{x\to a^+} f(x) = L$ and $\lim\limits_{x\to a^-} f(x) = L$ then $\lim\limits_{x\to a} f(x) = L$

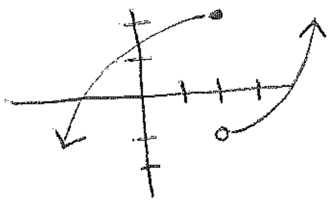
B. Solving Limits

1. Use Limit Theorem

ex1 $\lim\limits_{x\to 5^+} \dfrac{x^2+3}{x-4} = \dfrac{5^2+3}{5-4} = \boxed{28}$

2. Use Graphs

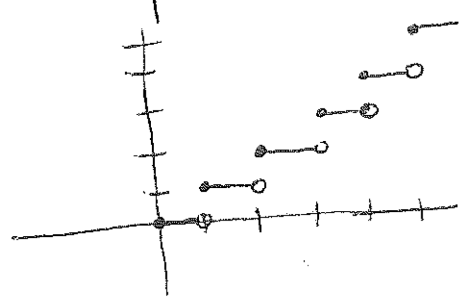
ex2 $\lim\limits_{x\to 2^+} f(x)$



$\lim\limits_{x\to 2^+} f(x) = \boxed{-1}$

3. Use tables

ex3 $\lim\limits_{x\to 4^-} [x]$



x	y
3.7	3
3.8	3
3.9	3
4	4

$\lim\limits_{x\to 4^-} [x] = \boxed{3}$

Note: $\lim\limits_{x\to 4} [x] = \text{DNE}$

4. Involving ∞

ex4 $\lim\limits_{x\to 0^+} \dfrac{1}{x} = \dfrac{1}{0} = +\infty = \text{DNE}$

close to but positive

this info helps us draw the graph

ex4 $\lim\limits_{x\to\infty} \dfrac{2x^2+5}{x^3+1} = \lim\limits_{x\to\infty} \dfrac{\cancel{x^3}\left(\frac{2}{x}+\frac{5}{x^3}\right)}{\cancel{x^3}\left(1+\frac{1}{x^3}\right)} = \dfrac{0+0}{1+0} = \dfrac{0}{1} = \boxed{0}$

5. Piecewise {New}

ex5 If $f(x) = \begin{cases} x-4 & x \geq 3 \\ -3x+2 & x < 3 \end{cases}$ ⓐ find $\lim\limits_{x\to 3} f(x)$

① $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} x-4 = -1$ ② $\lim\limits_{x\to 3^-} -3x+2 = -7$ $\therefore \lim\limits_{x\to 3} f(x) = \boxed{\text{DNE}}$

(b) find $\lim_{x\to 2} f(x) = \lim_{x\to 2} -3x+2 = -3(2)+2 = \boxed{-4}$

$\boxed{ex5}$ Find the value of K, such that $\lim_{x\to 4} f(x)$ exists

$$f(x) = \begin{cases} 3x+2 & x<4 \\ 5K+5 & x\geq 4 \end{cases}$$

for $\lim_{x\to 4} f(x)$ to exist ...

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^-} f(x)$$

$$\lim_{x\to 4^+} 5K+5 = \lim_{x\to 4^-} 3x+2$$

$$5K+5 = 3(4)+2$$

$$5K+5 = 14$$

$$5K = 9$$

$$\boxed{K = \frac{9}{5}}$$

$\boxed{ex5}$ If $f(x) = |x-7|$ find $\lim_{x\to 7} f(x)$

→ write as a piecewise

c.p. $x=7$

$$f(x) = \begin{cases} x-7 & x\geq 7 \\ -(x-7) & x<7 \end{cases} \qquad f(x) = \begin{cases} x-7 & x\geq 7 \\ -x+7 & x<7 \end{cases}$$

$$\lim_{x\to 7^+} x-7 = 7-7 = 0$$

$$\therefore \lim_{x\to 7} f(x) = \boxed{0}$$

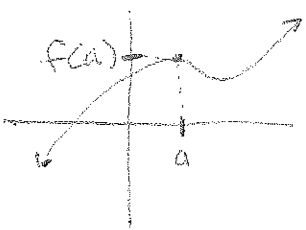
$$\lim_{x\to 7^-} -x+7 = -7+7 = 0$$

VI Continuity (1H)

Vague definition of continuity is if you can trace the graph of the function without lifting your pencil, the function is continuous.

Real definition of continuity is:

$f(x)$ is continuous at "a" if and only if:



① $f(a)$ exists

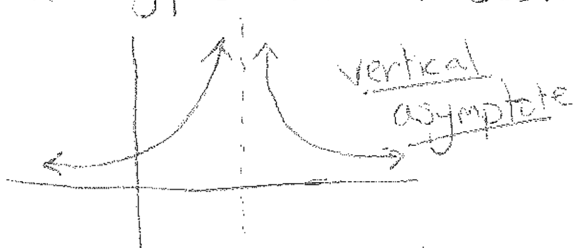
② $\lim_{x \to a} f(x)$ exists

③ $\lim_{x \to a} f(x) = f(a)$

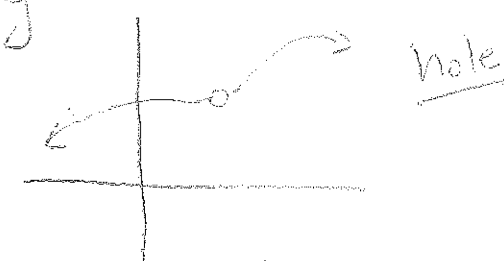
} all 3 must be true!

$f(x)$ is continuous on an interval if it is continuous at every number in the interval.

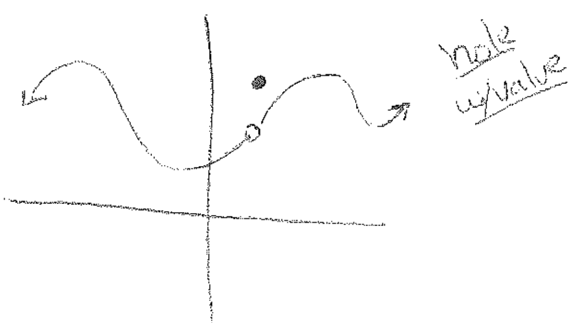
A. Types of Discontinuity



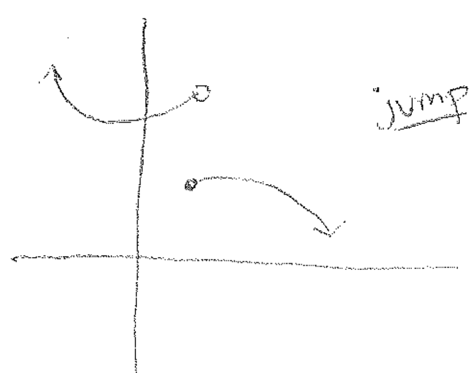
"infinite discontinuity"



"removable discontinuity"



"removable discontinuity"



"jump discontinuity"

B. Determining if an Equation is Continuous

$\boxed{\text{ex 1}}$ Is $f(x)$ continuous? means everywhere

$$f(x)=\begin{cases}\dfrac{x^2-3x-4}{x-4} & ; \; x\neq 4 \\ 2 & ; \; x=4\end{cases}$$

concerned at the domain

$x=4$ check "$a=4$"

Check all 3 parts

① $f(4)=2$ exists! :)

$\frac{0}{0}$ indeterminate form!

② $\lim\limits_{x\to 4} f(x) = \lim\limits_{x\to 4} \dfrac{x^2-3x-4}{x-4} = \lim\limits_{x\to 4} \dfrac{\cancel{(x-4)}(x+1)}{\cancel{x-4}} = \lim\limits_{x\to 4} x+1$

$= 4+1 = 5$ exists! :)

③ Does $f(4)=\lim\limits_{x\to 4} f(x)$? $2\neq 5$:(

$\therefore f(x)$ is not continuous

$\boxed{\text{ex 2}}$ Where is $f(x)=\dfrac{x^2-5x+6}{x-2}$ discontinuous?

concerned at domain, $x=2$ check $a=2$

① $f(2)=\dfrac{2^2-5(2)+6}{2-2}=\dfrac{0}{0} \therefore$ DNE

$\therefore f(x)$ is discontinuous at $x=2$

remember
$\lim\limits_{x\to a} f(x)=\frac{0}{0}$ indeterminate
$f(x)=\frac{0}{0}$ DNE

$\boxed{\text{ex 3}}$ Is $f(x)=\dfrac{x+3}{x+1}$ continuous on the interval

domain issue, $x=-1$

i. $(-2,4)$

ii. $(2,4)$

iii. $(-1,2)$

iiii $[-1,2)$

① $f(-1)=\dfrac{-1+3}{-1+1}=\dfrac{2}{0}$: DNE

$\therefore f(x)$ is not continuous at $x=-1$

i no b/c -1 is in the interval $(-2,4)$

ii yes

iii yes

iiii no b/c -1 is included in the interval ex4 For what values of c is f(x) continuous?

$$f(x)=\begin{cases} cx^2+2x & x<2 \\ x^2-cx & x\geq 2 \end{cases}$$

concerned at domain, $x=2$

① $f(2)=(2)^2-c(2)=4-2c$

② $\lim_{x\to 2} f(x)=$ two parts

i. $\lim_{x\to 2^+} f(x)=(2)^2-c(2)=4-2c$

ii $\lim_{x\to 2^-} f(x)=c(2)^2+2(2)=4c+4$

for $\lim_{x\to 2} f(x)$ to exist, $\lim_{x\to 2^+} f(x)=\lim_{x\to 2^-} f(x)$

$$4-2c=4c+4$$
$$0=6c$$
$$0=c$$

Does

③ $f(2)=\lim_{x\to 2^+} f(x)=\lim_{x\to 2^-} f(x)$

$4-2c=4-2c=4c+4$

$4-2(0)=4-2(0)=4(0)+4$ $\boxed{c=0}$

$4=4=4$:)

C. Additional Theorems

1. If f(x) is continuous from the right at "a" then

$$\lim_{x\to a^+} f(x)=f(a)$$

2. If f(x) is continuous from the left at "a" then

$$\lim_{x\to a^-} f(x)=f(a)$$

3. If f and g are continuous at "a" and c is a constant, then the following functions are also continuous at "a".

1. $f+g$ 2. $f-g$ 3. fg 4. cf

5. $\frac{f}{g}$ if $g(a) \neq 0$

4. Any polynomial is continuous everywhere, $\mathbb{R}$ or $(-\infty, \infty)$

5. Any rational function is continuous wherever it is defined { continuous over its domain }

→ also works with trig functions, root functions, exponential functions & log functions.