3E

Mrs. Marcelino
Unit 3

Name: ____________
Date: ________ Period: _____

# Practice: Related Rates (3E)

| | |
|---|---|
| 3E | Solve related rates problems in a variety of pure and applied contexts (e.g. balloon, "car", ladder, shadow, cone, kite) and explain how implicit differentiation is involved in finding the solution. |

L2 A cube is expanding with time. How is the rate at which the volume increases related to the rate at which the length of a side increases?

L2 2. The volume of a rectangular box is $V = xyz$. Given that each side expands at a constant rate of 10 cm/min, find the rate at which the volume is expanding when $x = 1$ cm, $y = 2$ cm, and $z = 3$ cm.

L2 3. A plate in the shape of an equilateral triangle expands with time. The length of a side increases at a constant rate of 2 cm/h. At what rate is the area increasing when a side is 8 cm?

L3 4. In Problem 3 at what rate is the area increasing at the instant when the area is $\sqrt{75}$ cm$^2$?

L2 5. A rectangle expands with time. The diagonal of the rectangle increases at a rate of 1 in/h and the length increases at a rate of $\frac{1}{4}$ in/h. How fast is its width increasing when the width is 6 in and the length is 8 in?

L2 The lengths of the sides of a cube increase at a rate of 5 cm/h. At what rate does the length of the diagonal of the cube increase?

A boat is sailing toward the vertical cliff shown in Figure 3.16. How are the rates at which $x$, $s$, and $\theta$ change related?
L3

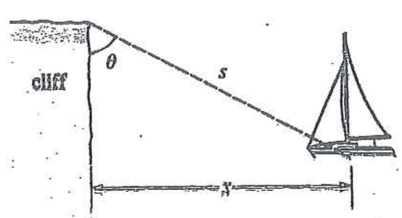



FIGURE 3.16

L4 8. The total resistance $R$ in a parallel circuit that contains two resistors of resistances $R_1$ and $R_2$ is given by $1/R = 1/R_1 + 1/R_2$. Each resistance changes with time. How are $dR/dt$, $dR_1/dt$, and $dR_2/dt$ related?

L2 9. A bug crawls along the graph of $y = x^2 + 4x + 1$, where $x$ and $y$ are measured in centimeters. If the abscissa $x$ changes at a constant rate of 3 cm/min, how fast is the ordinate changing at the point (2, 13)?
x coordinate
y coordinate

L3 10. In Problem 9 how fast is the ordinate changing when the bug is 6 cm above the x-axis?

L2 11. A particle moves on the graph of $y^2 = x + 1$ so that $dx/dt = 4x + 4$. What is $dy/dt$ when $x = 8$?

L3 12. A particle in continuous motion moves on the graph of $4y = x^2 + x$. Find the point on the graph at which the rate of change of the abscissa and the rate of change of the ordinate are the same.

L3 13. The x-coordinate of the point $P$ shown in Figure 3.17 increases at a rate of $\frac{1}{3}$ cm/h. How fast is the area of the right triangle $OPA$ increasing when $P$ has coordinates (8, 2)?

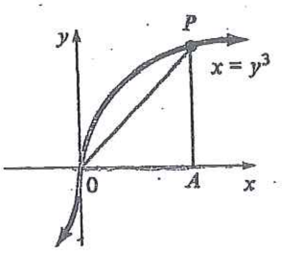



FIGURE 3.17

L2 14. A suitcase is carried up the conveyor belt shown in Figure 3.18 at a rate of 2 ft/s. How fast is the suitcase rising?

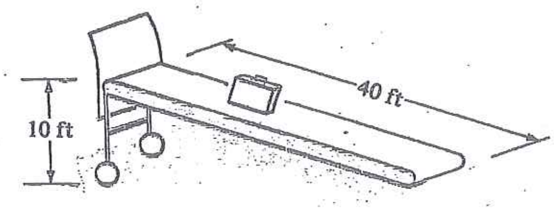



FIGURE 3.18

L4 15. In the adiabatic expansion of air, pressure $P$ and volume $V$ are related by $PV^{1.4} = k$, where $k$ is a constant. At a certain instant the pressure is 100 lb/in$^2$ and the volume is 32 in$^3$. At what rate is the pressure changing at that instant if the volume is decreasing at a rate of 2 in$^3$/s?

L2 16. A stone dropped into a still pond causes a circular wave. Assume the radius of the wave expands at a constant rate of 2 ft/s.

(a) How fast does the diameter increase?

(b) How fast does the circumference increase?

(c) How fast does the area expand when the radius is 3 ft?

(d) How fast does the area expand when the area is $8\pi$ ft$^2$?

L2 17. An oil tank in the shape of a right circular cylinder of radius 8 m is being filled at a constant rate of 10 m$^3$/min. How fast is the level of the oil rising?

L2 18. A water tank in the shape of a right circular cylinder of diameter 40 ft is being drained so that the level of the water decreases at a constant rate of $\frac{3}{2}$ ft/min. How fast is the volume of the water decreasing?

L3 19. Assume that a cube of ice melts in such a manner that it always retains its cubical shape. If the volume of the cube decreases at a rate of $\frac{1}{4}$ in$^3$/min, how fast is the surface area of the cube changing when the surface area is 54 in$^2$?

20. As shown in Figure 3.19, a 5-ft-wide rectangular water tank is divided into two tanks by a partition that moves in the direction indicated at a rate of 1 in/min as water is pumped into the front tank at a rate of 1 ft³/min. At what rate is the level of the water changing when the volume of the water in the front tank is 40 ft³ and $x = 4$ ft? Is the level of the water rising or falling at that instant?

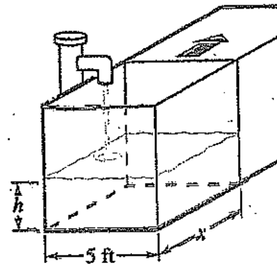


Figure 3.19

L4
21. Each vertical end of a 20-ft-long water trough is an equilateral triangle with vertex down. Water is being pumped in at a constant rate of 4 ft³/min.

(a) How fast is the level $h$ of the water rising when the water is 1 ft deep?

(b) If $h_0$ is the initial depth of water in the trough, show that

$$\frac{dh}{dt} = \frac{\sqrt{3}}{10}\left(h_0^2 + \frac{\sqrt{3}}{5}t\right)^{-1/2}$$

[*Hint:* Consider the difference in volumes after $t$ minutes.]

(c) If $h_0 = \frac{1}{2}$ ft and the height of the triangular end is 5 ft, determine the time when the trough is full. How fast is the level of the water rising when the trough is full?

L4
22. A water trough with vertical ends in the form of isosceles trapezoids has dimensions as shown in Figure 3.20. If water is pumped in at a constant rate of $\frac{1}{2}$ m³/s, how fast is the level of the water rising when the water is $\frac{1}{4}$ m deep?

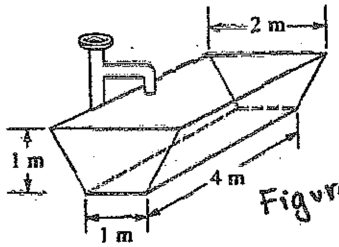


Figure 3.20

2
23. Water leaks out the bottom of the conical tank shown in Figure 3.21 at a constant rate of 1 ft³/min.

(a) At what rate is the level of the water changing when the water is 6 ft deep?

(b) At what rate is the radius of the water changing when the water is 6 ft deep?

L3 (c) Assume the tank was full at $t = 0$. At what rate is the radius of the water changing at $t = 6$ min?

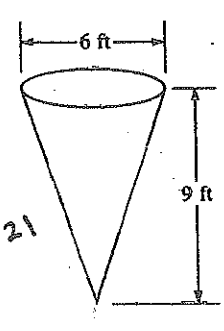


Figure 3.21

L3
24. A 5-ft-tall person walks away from a 20-ft-tall street lamp at a constant rate of 3 ft/s. See Figure 3.12(b).

(a) At what rate is the length of the person's shadow increasing?

(b) At what rate is the tip of the shadow moving away from the base of the street lamp?

L2
25. A 15-ft ladder is leaning against a wall of a house. The bottom of the ladder is pulled away from the base of the wall at a constant rate of 2 ft/min. At what rate is the top of the ladder sliding down the wall when the bottom of the ladder is 5 ft from the wall?

L2
26. A kite string is paid out at a constant rate of 3 ft/s. If the wind carries the kite horizontally at an altitude of 200 ft, how fast is the kite moving when 400 ft of string have been paid out?

L2 27. A plane flying parallel to level ground at a constant rate of 600 mi/h approaches a radar station. If the altitude of the plane is 2 mi, how fast is the distance between the plane and the radar station decreasing when the horizontal distance between them is 1.5 mi? See Figure 3.22.

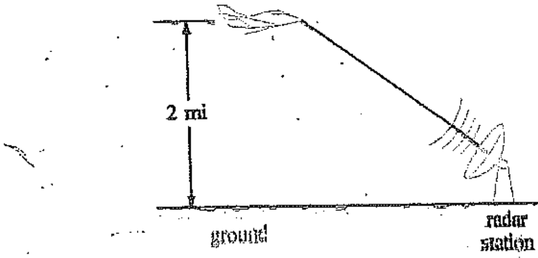


Figure 3.22

L3
28. In Problem 27, at the point directly above the radar station, the plane goes into a 30° climb while retaining the same speed. How fast is the distance between the plane and radar station increasing 1 min later? [*Hint:* Review the law of cosines.]

L2
29. A plane at an altitude of 4 km passes directly over a tracking telescope on the ground. When the angle of elevation is 60°, it is observed that this angle is decreasing at a rate of 30 deg/min. How fast is the plane traveling?

L3 30. A rocket is traveling at a constant rate of 1000 mi/h at an angle of 60° to the horizontal. See Figure 3.23.

(a) At what rate is its altitude increasing?

(b) What is the ground speed of the rocket?

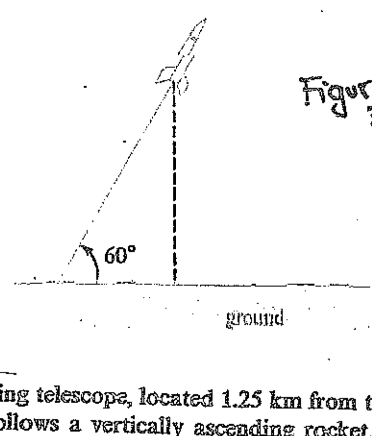


Figure 3.23

L3 31. A tracking telescope, located 1.25 km from the point of launching, follows a vertically ascending rocket. When the angle of elevation is 60°, the rate at which the angle is increasing is 3 deg/s. At what rate is the rocket moving at that instant?

L4 32. The volume $V$ between two concentric spheres is expanding. The radius of the outer sphere increases at a constant rate of 2 m/h, whereas the radius of the inner sphere decreases at a constant rate of $\frac{1}{2}$ m/h. At what rate is $V$ changing when the outer radius is 3 m and the inner radius is 1 m?

L4 33. Many spherical objects such as raindrops, snowballs, and mothballs evaporate at a rate proportional to their surface areas. In this case show that the radius of the object decreases at a constant rate.

proportional: $y = cx$ inversely prop: $y = \frac{c}{x}$

L3 34. If the rate at which the volume of a sphere changes is constant, show that the rate at which its surface area changes is inversely proportional to the radius.

L3 35. Two tankers depart from the same floating oil terminal. One tanker sails east at noon at a rate of 10 knots. (1 knot = 1 nautical mile/hour. A nautical mile is 6080 ft or 1.15 statute miles.) The other tanker sails north at 1:00 P.M. at a rate of 15 knots. At 2:00 P.M. at what rate is the distance between the two ships changing?

L3 36. At 8:00 A.M. ship $S_1$ is 20 km due north of ship $S_2$. Ship $S_1$ sails south at a rate of 9 km/h and ship $S_2$ sails west at a rate of 12 km/h. At 9:20 A.M. at what rate is the distance between the two ships changing?

L3 37. Sand flows from the top half of the conical hourglass shown in Figure 3.24 to the bottom half at a constant rate of 4 cm³/s. At any time, assume that the height of the sand is $h$ and that the sand has the shape of a frustum of a cone. Express the rate at which $h$ increases in terms of $h$.

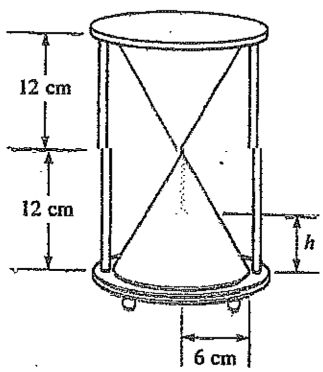



FIGURE 3.24

L4 38. The Ferris wheel shown in Figure 3.25 revolves counterclockwise once every 2 min. How fast is a passenger rising at the instant when she is 64 ft above the ground? How fast is she moving horizontally at the same instant?

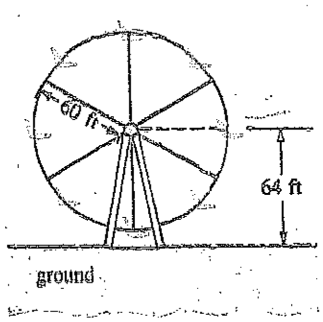



FIGURE 3.25

L4 39. Suppose the Ferris wheel in Problem 38 is equipped with bidirectional colored spotlights fixed at various points on its circumference. Consider the spotlight located at point $P$ in Figure 3.26. If the light beams are tangent to the wheel at point $P$, at what rate is the spot $Q$ on the ground moving away from point $R$ when $\theta = \pi/4$?

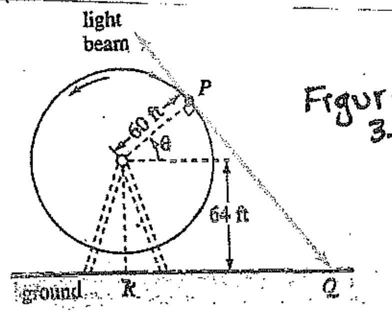



Figure 3.26

L4 40. In physics the momentum $p$ of a body of mass $m$ that moves in a straight line with velocity $v$ is given by $p = mv$. An airplane of mass $10^5$ kg flies in a straight line while ice builds up on the leading edges of its wings at a constant rate of 30 kg/h. See Figure 3.27.

(a) At what rate is the momentum of the plane changing if it is flying at a constant rate of 800 km/h?

(b) At what rate is the momentum of the plane changing at $t = 1$ h if at that instant its velocity is 750 km/h and is increasing at a rate of 20 km/h²?

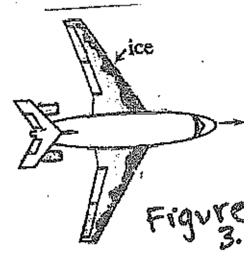



Figure 3.27

L4 41. A study of crayfish (*Orconectes virilis*) indicates that the carapace length $C$ is related to the total length $T$ according to the formula $C = 0.493T - 0.913$, where $C$ and $T$ are measured in millimeters. See Figure 3.28.

(a) As the crayfish grows, does the ratio $R$ of the carapace length to the total length increase or decrease?

(b) If the crayfish grows in length at the rate of 1 mm per day, at what rate is the ratio of the carapace to the total length changing when the carapace is one-third of the total length?

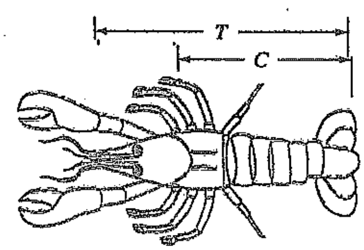
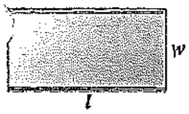



FIGURE 3.28

L4 42. According to allometric studies, brain weight $E$ in fish is related to body weight $P$ by $E = 0.007P^{2/3}$, and body weight is related to body length $L$ by $P = 0.12L^{2.53}$, where $E$ and $P$ are measured in grams and $L$ is measured in centimeters. Suppose that the length of a certain species of fish evolved at a constant rate from 10 cm to 18 cm over the course of 20 million years.

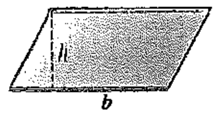
ı $A$; circumference $C$; volume $V$; surface area $S$

**RECTANGLE**



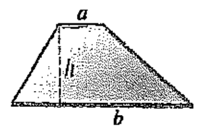

$A = lw \qquad C = 2l + 2w$

**PARALLELOGRAM**



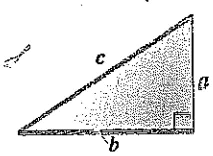

$A = bh$

**TRAPEZOID**



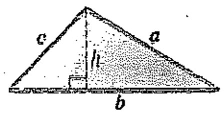

$A = \frac{1}{2}(a + b)h$

**RIGHT TRIANGLE**



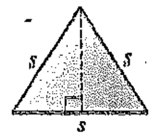

Pythagorean Theorem:
$c^2 = a^2 + b^2$

**TRIANGLE**




$A = \frac{1}{2}bh \qquad C = a + b + c$

**EQUILATERAL TRIANGLE**

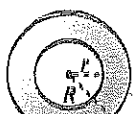



$h = \frac{\sqrt{3}}{2}s \qquad A = \frac{\sqrt{3}}{4}s^2$

**CIRCLE**

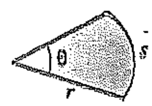

$A = \pi r^2 \qquad C = 2\pi r$

**CIRCULAR RING**




$A = \pi(R^2 - r^2)$

**CIRCULAR SECTOR**

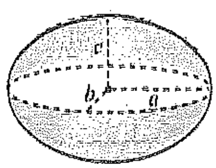



$A = \frac{1}{2}r^2\theta \qquad s = r\theta$

**ELLIPSE**


$A = \pi ab$

**ELLIPSOID**




$V = \frac{4}{3}\pi abc$

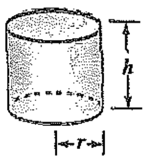
**SPHERE**


$V = \frac{4}{3}\pi r^3 \qquad S = 4\pi r^2$

**RIGHT CYLINDER**

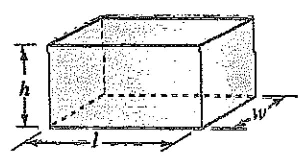

$V = Bh$, $B$ area of base

**RIGHT CIRCULAR CYLINDER**



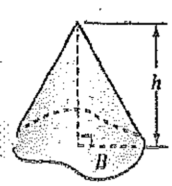

$V = \pi r^2 h \qquad S = 2\pi rh$

**RECTANGULAR PARALLELEPIPED**



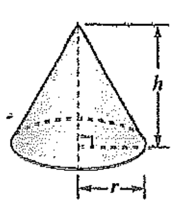

$V = lwh \qquad S = 2(hl + lw + hw)$

**CONE**



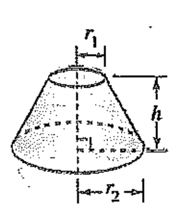

$V = \frac{1}{3}Bh$, $B$ area of base

**RIGHT CIRCULAR CONE**




$V = \frac{1}{3}\pi r^2 h \qquad S = \pi r\sqrt{r^2 + h^2}$

**FRUSTUM OF A CONE**




$V = \frac{1}{3}\pi h(r_1^2 + r_1 r_2 + r_2^2)$

Answers to odd:

1. $\frac{dV}{dt} = 3x^2\frac{dx}{dt}$ 3. $8\sqrt{3}$ cm²/h
5. $\frac{4}{3}$ in/h 7. $s\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{ds}{dt} = \frac{dx}{dt}$
9. 24 cm/min 11. $-6$ or 6 13. $\frac{4}{9}$ cm²/h
15. $\frac{35}{4}$ lb/in²/s 17. $5/(32\pi)$ m/min
19. $-\frac{1}{3}$ in²/min
21. $\sqrt{3}/10$ ft/min; 71.45 min; 0.035 ft/min
23. $-1/(4\pi)$ ft/min; $-1/(12\pi)$ ft/min; approximately $-0.0124$ ft/min
25. $-\sqrt{2}/2$ ft/min 27. $-360$ mi/h
29. $8\pi/9$ km/min 31. $\pi/12$ km/s
35. 17 knots 37. $\frac{dh}{dt} = \frac{16}{\pi(12 - h)^2}$
39. 668.7 ft/min
41. Increase approximately 2.8% per day

1. Let $V$ be the volume and $x$ the length of a side. Then $V = x^3$ and $\frac{dV}{dt} = 3x^2\frac{dx}{dt}$.

2. From $V = xyz$ we find

$$\frac{dV}{dt} = x\frac{d}{dt}(yz) + yz\frac{dx}{dt} = x\left(y\frac{dz}{dt} + z\frac{dy}{dt}\right) + yz\frac{dx}{dt} = xy\frac{dz}{dt} + xz\frac{dy}{dt} + yz\frac{dx}{dt}.$$

Using the given sides and rates, we obtain $\frac{dV}{dt} = 1\cdot 2\cdot 10 + 1\cdot 3\cdot 10 + 2\cdot 3\cdot 10 = 110\,\text{cm}^3/\text{s}$.

3. Let $A$ be the area and let $x$ be the length of a side. Then $A = \frac{\sqrt{3}\,x^2}{4}$ and $\frac{dA}{dt} = \frac{\sqrt{3}}{2}x\frac{dx}{dt}$. When $\frac{dx}{dt} = 2$ and $x = 8$ we have $\frac{dA}{dt} = \frac{\sqrt{3}}{2}(8)(2) = 8\sqrt{3}\,\text{cm}^2/\text{h}$.

4. From Problem 3 we have $A = \frac{\sqrt{3}\,x^2}{4}$ and $\frac{dA}{dt} = \frac{\sqrt{3}}{2}x\frac{dx}{dt}$. Since $A = \sqrt{75}$, we have $x^2 = 20$ and $x = 2\sqrt{5}$. When $\frac{dx}{dt} = 2\,\text{cm/h}$ and $x = 2\sqrt{5}\,\text{cm}$, $\frac{dA}{dt} = \frac{\sqrt{3}}{2}(2\sqrt{5})(2) = 2\sqrt{15}\,\text{cm}^2/\text{h}$.

5. Let $x$ be the length, $y$ the width and $s$ the diagonal of the rectangle. Then $s^2 = x^2 + y^2$ or $y^2 = s^2 - x^2$, and $2y\frac{dy}{dt} = 2s\frac{ds}{dt} - 2x\frac{dx}{dt}$ or $\frac{dy}{dt} = \frac{s}{y}\frac{ds}{dt} - \frac{x}{y}\frac{dx}{dt}$. When $x = 8\,\text{in}$ and $y = 6\,\text{in}$, $s = 10\,\text{in}$. Then $\frac{dy}{dt} = \frac{10}{6}(1) - \frac{8}{6}\left(\frac{1}{4}\right) = \frac{4}{3}\,\text{in/h}$.

6. Let $x$ be the side of a cube and $s$ the diagonal. Then $s^2 = 3x^2$ and $2s\frac{ds}{dt} = 6x\frac{dx}{dt}$ or $\frac{ds}{dt} = 3\frac{x}{z}\frac{dx}{dt}$. When $dx/dt = 5\,\text{cm/h}$, $\frac{ds}{dt} = 3\frac{x}{\sqrt{3}\,x}(5) = 5\sqrt{3}\,\text{cm/h}$.

7. $\sin\theta = x/s$ or $x = s\sin\theta$. Differentiating with respect to $t$ gives $\frac{dx}{dt} = s\frac{d}{dt}\sin\theta + (\sin\theta)\frac{ds}{dt} = s\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{ds}{dt}$.

8. $R^{-1} = R_1^{-1} + R_2^{-1}$. Differentiating with respect to $t$ gives

$$-R^{-2}\frac{dR}{dt} = -R_1^{-2}\frac{dR_1}{dt} - R_2^{-2}\frac{dR_2}{dt} \quad\text{and}\quad \frac{1}{R_2}\frac{dR}{dt} = \frac{1}{R_1^2}\frac{dR_1}{dt} + \frac{1}{R_2^2}\frac{dR_2}{dt}$$

so $\frac{dR}{dt} = \frac{R^2}{R_1^2}\frac{dR_1}{dt} + \frac{R^2}{R_2^2}\frac{dR_2}{dt}$.

9. $\frac{dy}{dt} = 2x\frac{dx}{dt} + 4\frac{dx}{dt}$. When $x = 2\,\text{cm}$ and $\frac{dx}{dt} = 3\,\text{cm/min}$, $\frac{dy}{dt} = 2\cdot 2\cdot 3 + 4\cdot 3 = 24\,\text{cm/min}$.

10. When $y = 6$, we have $x^2 + 4x + 1 = 6$, $x^2 + 4x - 5 = 0$, and $(x+5)(x-1) = 0$. Thus $x = -5, 1$. From Problem 9 we have $\frac{dy}{dt} = 2x\frac{dx}{dt} + 4\frac{dx}{dt}$, so for $\frac{dx}{dt} = 3$,

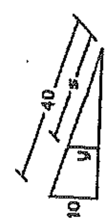
$$\left.\frac{dy}{dt}\right|_{x=1} = 2\cdot 1\cdot 3 + 4\cdot 3 = 18\,\text{cm/min} \quad\text{and}\quad \left.\frac{dy}{dt}\right|_{x=-5} = 2(-5)\cdot 3 + 4\cdot 3 = -18\,\text{cm/min}.$$

11. $2y\frac{dy}{dt} = \frac{dx}{dt}$; $\frac{dy}{dt} = \frac{dx/dt}{2y}$. From $y^2 = x + 1$ we see that for $x = 8$, $y = \pm 3$. Since $\frac{dx}{dt} = 4x + 4$, we have $\frac{dy}{dt} = \frac{4x+4}{2y}$. Thus $\left.\frac{dy}{dt}\right|_{y=3} = \frac{4\cdot 8 + 4}{2\cdot 3} = 6$ and $\left.\frac{dy}{dt}\right|_{y=-3} = \frac{4\cdot 8 + 4}{2(-3)} = -6$.

12. $4\frac{dy}{dt} = 2x\frac{dx}{dt} + \frac{dx}{dt}$. Since $\frac{dx}{dt} = \frac{dy}{dt}$, we cancel the derivatives and obtain $4 = 2x + 1$ or $x = 3/2$. From $4y = x^2 + x$ we see that for $x = 3/2$, $y = 15/16$. Hence the point on the graph is $(3/2, 15/16)$.

13. If $T$ is the area of the triangle then $T = \frac{1}{2}xy = \frac{1}{2}x^{4/3}$ and $\frac{dT}{dt} = \frac{2}{3}x^{1/3}\frac{dx}{dt}$. When $x = 8$, then $\frac{dT}{dt} = \frac{2}{3}(8)^{1/3}\left(\frac{1}{3}\right) = \frac{4}{9}\,\text{cm}^2/\text{h}$.

14. Using similar triangles $y/s = 10/40$ or $y = s/4$. Then $\frac{dy}{dt} = \frac{1}{4}\frac{ds}{dt}$ and since $\frac{ds}{dt} = 2\,\text{ft/s}$, $\frac{dy}{dt} = \frac{1}{4}(2) = \frac{1}{2}\,\text{ft/s}$.




15. From $PV^{1.4} = k$ we obtain $P\frac{d}{dt}V^{1.4} + V^{1.4}\frac{dP}{dt} = 0$ and $P(1.4V^{0.4}\frac{dV}{dt}) + V^{1.4}\frac{dP}{dt} = 0$ so $\frac{dP}{dt} = -\frac{1.4PV^{0.4}}{V^{1.4}}\frac{dV}{dt}$. When $P = 100\,\text{lb/in}^2$, $V = 32\,\text{in}^3$ and $\frac{dV}{dt} = -2\,\text{in}^3/\text{s}$, so $\frac{dP}{dt} = -\frac{1.4(100)(32)^{0.4}}{(32)^{1.4}}(-2) = 8.75\,\text{lb/in}^2/\text{s}$.

**(a)** Since $D = 2r$, we have $\dfrac{dD}{dt} = 2\dfrac{dr}{dt} = 2\cdot 2 = 4$ ft/s.

**(b)** Since $C = 2\pi r$, we have $\dfrac{dC}{dt} = 2\pi\dfrac{dr}{dt} = 2\pi\cdot 2 = 4\pi$ ft/s.

**(c)** Since $A = \pi r^2$ we have $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$. When $\dfrac{dr}{dt} = 2$ ft/s and $r = 3$ ft, $\dfrac{dA}{dt} = 2\pi\cdot 3\cdot 2 = 12\pi$ ft$^2$/s.

**(d)** Since $A = \pi r^2 = 8\pi$ we have $r = 2\sqrt{2}$. When $\dfrac{dr}{dt} = 2$ ft/s and $r = 2\sqrt{2}$ ft, $\dfrac{dA}{dt} = 2\pi(2\sqrt{2})(2) = 8\pi\sqrt{2}$ ft$^2$/s.

$V = \pi r^2 h$. Since $r$ is a constant, differentiating with respect to $t$ gives $\dfrac{dV}{dt} = \pi r^2 h\dfrac{dh}{dt}$ or $\dfrac{dh}{dt} = \dfrac{1}{\pi r^2}\dfrac{dV}{dt}$. When $r = 8$ m and $\dfrac{dV}{dt} = 10\,\text{m}^3/\text{min}$, the oil level rises at a rate of $\dfrac{dh}{dt} = \dfrac{10}{\pi(8)^2} = \dfrac{5}{32\pi}$ m/min.

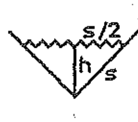
$V = \pi r^2 h$. Since $r$ is a constant, differentiating with respect to $t$ gives $\dfrac{dV}{dt} = \pi r^2 h\dfrac{dh}{dt}$. When $r = 40/2 = 20$ ft and $\dfrac{dh}{dt} = -3/2$ ft/min, $\dfrac{dV}{dt} = \pi(20)^2(-3/2) = -600\pi$ ft$^3$/min. Thus the volume is decreasing at a rate of $600\pi$ ft$^3$/min.

$V = x^3$, $\dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt}$. The surface area of the cube is $S = 6x^2$, so when $S = 54$, $x = 3$. Now when $\dfrac{dV}{dt} = -\dfrac{1}{4}$ we have $-\dfrac{1}{4} = 3(3)^2\dfrac{dx}{dt}$ and $\dfrac{dx}{dt} = -\dfrac{1}{108}$. From $\dfrac{dS}{dt} = 12x\dfrac{dx}{dt}$ we use $x = 3$ and $\dfrac{dx}{dt} = -\dfrac{1}{108}$ to compute $\dfrac{dS}{dt} = 12(3)\left(-\dfrac{1}{108}\right) = -\dfrac{1}{3}$ in$^2$/min.

The volume of water is $V = 5xh$, so $\dfrac{dV}{dt} = 5x\dfrac{dh}{dt} + 5h\dfrac{dx}{dt}$. We are given $\dfrac{dV}{dt} = 1$, $\dfrac{dx}{dt} = \dfrac{1}{12}$, and $x = 4$. From $V = 40$ and $x = 4$ we see that $h = 2$. Then $1 = 5(4)\dfrac{dh}{dt} + 5(2)\dfrac{1}{12}$ and

$\dfrac{dh}{dt} = \dfrac{1}{120}$ ft/min $= \dfrac{1}{10}$ in/min. The water is rising at this instant.

1. **(a)** From the Pythagorean Theorem, $s^2 = h^2 + (s/2)^2$ or $s = \dfrac{2h}{\sqrt{3}}$. The volume of the water is $V = \dfrac{1}{2}sh(20) = 10\dfrac{2h}{\sqrt{3}}h = \dfrac{20h^2}{\sqrt{3}}$. Differentiating with respect to $t$ gives $\dfrac{dV}{dt} = \dfrac{40h}{\sqrt{3}}\dfrac{dh}{dt}$ so $\dfrac{dh}{dt} = \dfrac{\sqrt{3}}{40h}\dfrac{dV}{dt}$. When $h = 1$ ft and $\dfrac{dV}{dt} = 4$ ft$^3$/min, the rate at which the water level rises is $\dfrac{dh}{dt} = \dfrac{\sqrt{3}}{10}$ ft/min.




**(b)** From part (a) we see that the initial volume of water is $V_0 = \dfrac{20h_0^2}{\sqrt{3}}$. At time $t$ the volume of water is $V = 4t + V_0 = 4t + \dfrac{20h_0^2}{\sqrt{3}}$. In terms of $h$ we saw in part (a) that $V = \dfrac{20h^2}{\sqrt{3}}$. Thus $\dfrac{20h^2}{\sqrt{3}} = 4t + \dfrac{20h_0^2}{\sqrt{3}}$. Solving for $h$ and differentiating, we find

$$h = \sqrt{\frac{\sqrt{3}}{5}t + h_0^2} \quad\text{and}\quad \frac{dh}{dt} = \frac{1}{2}\left(\frac{\sqrt{3}}{5}t + h_0^2\right)^{-1/2}\frac{\sqrt{3}}{5} = \frac{\sqrt{3}}{10}\left(\frac{\sqrt{3}}{5}t + h_0^2\right)^{-1/2}.$$

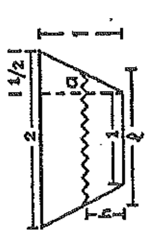
**(c)** Setting $h = 5$ and $h_0 = \frac{1}{2}$ in part (b) we have $5 = \sqrt{\dfrac{\sqrt{3}}{5}t + \dfrac{1}{4}}$ or $t = \dfrac{165\sqrt{3}}{4} \approx 71.45$ min.

The rate at which the water is rising when $t = \dfrac{165\sqrt{3}}{4}$ is

$$\left.\frac{dh}{dt}\right|_{t=165\sqrt{3}/4} = \frac{\sqrt{3}}{10}\left(\frac{\sqrt{3}}{5}\cdot\frac{165\sqrt{3}}{4} + \frac{1}{4}\right)^{-1/2} = \frac{\sqrt{3}}{50} \approx 0.035 \text{ ft/min}.$$

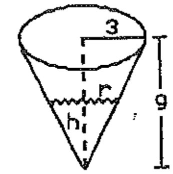
22. Since the lengths of corresponding sides in similar triangles are proportional, $a/h = 1/2$ and $\ell = 1 + 2a = 1 + h$. The volume of water is $V = \dfrac{h(1+\ell)}{2}(4) = 2h(2+h) = 4h + 2h^2$. Differentiating gives

$$\frac{dV}{dt} = (4+4h)\frac{dh}{dt} \quad\text{or}\quad \frac{dh}{dt} = \frac{1}{4+4h}\frac{dV}{dt}.$$




When $h = \dfrac{1}{4}$ m and $\dfrac{dV}{dt} = \dfrac{1}{2}$ m$^3$/s, the rate at which the water level rises is $\dfrac{dh}{dt} = \dfrac{1}{4 + 4(1/4)}\cdot\dfrac{1}{2} = \dfrac{1}{10}$ m/s.

**23.** (a) Since the lengths of corresponding sides in similar triangles are proportional, $\frac{h}{9} = \frac{r}{3}$ or $h = 3r$. The volume of water is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{1}{27}\pi h^3$. Differentiating gives



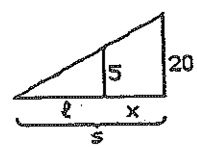

$$\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt} \quad \text{or} \quad \frac{dh}{dt} = \frac{9}{\pi h^2}\frac{dV}{dt}.$$

We are given $\frac{dV}{dt} = -1$. When $h = 6$ the water level is changing at a rate of $\frac{dh}{dt} = \frac{9}{\pi(36)}(-1) = -\frac{1}{4\pi}$ ft/min.

(b) From $h = 3r$ we have $\frac{dh}{dt} = 3\frac{dr}{dt}$, so when $h = 6$ the radius of watr is changing at a rate of $\frac{dr}{dt} = \frac{1}{3}\frac{dh}{dt} = \frac{1}{3}\left(-\frac{1}{4\pi}\right) = -\frac{1}{12\pi}$ ft/min.

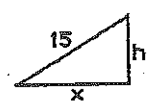
(c) The initial volume of water is $V_0 = \frac{1}{3}\pi(3)^2(9) = 27\pi\,\text{ft}^3$. At time $t$ the volume of water is $V(t) = V_0 - t = 27\pi - t$. We also have from part (a) that $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2(3r) = \pi r^3$. Thus, $\pi r^3 = 27\pi - t$ or $r = \left(27 - \frac{t}{\pi}\right)^{1/3}$. Then $\frac{dr}{dt} = -\frac{1}{3\pi}\left(27 - \frac{t}{\pi}\right)^{-2/3}$ and $\left.\frac{dr}{dt}\right|_{t=6} = -\frac{1}{3\pi(27 - 6/\pi)^{2/3}} \approx -0.0124$ ft/min.

**24.** (a) Since the lengths of corresponding sides in similar triangles are proportional, $\frac{5}{20} = \frac{\ell}{\ell + x}$ or $\ell = x/3$. When $\frac{dx}{dt} = 3\,\text{ft/s}$, differentiating gives $\frac{d\ell}{dt} = \frac{1}{3}\frac{dx}{dt} = \frac{1}{3}(3) = 1$ ft/s.




(b) Differentiating $s = \ell + x$ gives $\frac{ds}{dt} = \frac{d\ell}{dt} + \frac{dx}{dt}$. Since $\frac{dx}{dt} = 3$ ft/s and from (a) $\frac{d\ell}{dt} = 1\,\text{ft/s}$, the tip of the shadow is moving at a rate of $\frac{ds}{dt} = 1 + 3 = 4\,\text{ft/s}$.

**25.** From the Pythagorean Theorem, $15^2 = h^2 + x^2$. Differentiating gives




$$0 = 2h\frac{dh}{dt} + 2x\frac{dx}{dt} \quad \text{or} \quad \frac{dh}{dt} = -\frac{x}{h}\frac{dx}{dt}.$$

When $x = 5\,\text{ft}$, $h = 10\sqrt{2}$ ft, and $\frac{dx}{dt} = 2\,\text{ft/min}$, we have

$$\frac{dh}{dt} = -\frac{5}{10\sqrt{2}}(2) = -\frac{1}{\sqrt{2}}\ \text{ft/min}.$$

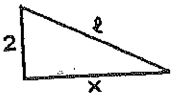
**26.** From the Pythagorean Theorem, $\ell^2 = x^2 + 200^2$. Differentiating gives


$$2\ell\frac{d\ell}{dt} = 2x\frac{dx}{dt} \quad \text{or} \quad \frac{dx}{dt} = \frac{\ell}{x}\frac{d\ell}{dt}.$$

We are given $\frac{d\ell}{dt} = 3\,\text{ft/s}$, and when $\ell = 400$ ft we have $x^2 = 400^2 - 200^2$ or $x = 200\sqrt{3}$ ft. Then the kite moves at a rate of

$$\frac{dx}{dt} = \frac{400}{200\sqrt{3}}(3) = 2\sqrt{3}\ \text{ft/s}.$$

27. From the Pythagorean Theorem, $\ell^2 = x^2 + 4$. Differentiating gives



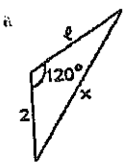

$$2\ell \frac{d\ell}{dt} = 2x \frac{dx}{dt} \quad \text{or} \quad \frac{d\ell}{dt} = \frac{x}{\ell}\frac{dx}{dt}.$$

We are given $\frac{dx}{dt} = -600\,\text{mi/h}$, and when $x = 1.5\,\text{mi}$, $\ell = 2.5\,\text{mi}$. Then

$$\frac{d\ell}{dt} = \frac{1.5}{2.5}(-600) = -360 \text{ mi/h}.$$

Thus the distance is decreasing at a rate of 360 mi/h.

28. From the law of cosines, $x^2 = 2^2 + \ell^2 - 2(2)\ell\cos 120° = 4 + \ell^2 + 2\ell$. Differentiating gives

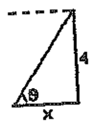



$$2x\frac{dx}{dt} = (2\ell + 2)\frac{d\ell}{dt} \quad \text{or} \quad \frac{dx}{dt} = \frac{\ell+1}{x}\frac{d\ell}{dt}.$$

We are given $\frac{d\ell}{dt} = 600\,\text{mi/h}$. After one minute the plane has travelled 10 miles. When $\ell = 10\,\text{mi}$, $x^2 = 4 + 10^2 + 2(10)$ and $x = 2\sqrt{31}$ mi. Thus,

$$\frac{dx}{dt} = \frac{10+1}{2\sqrt{31}}(600) = \frac{3300}{\sqrt{31}} \approx 592.70 \text{ mi/h}.$$

29. Differentiating $x = 4\cot\theta$, we obtain




$$\frac{dx}{dt} = -4\csc^2\theta\frac{d\theta}{dt}.$$

Converting 30° to $\pi/6$ radians, we are given $\frac{d\theta}{dt} = -\frac{\pi}{6}$. Thus, when $\theta = 60°$,

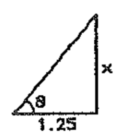
$$\frac{dx}{dt} = -4(\csc^2 60°)\left(-\frac{\pi}{6}\right) = \frac{8\pi}{9} \approx 2.79 \text{ km/min}.$$

30. Let $y$ be the altitude of the rocket, $x$ the distance along the ground from the point of launch, and $s$ the distance the rocket has travelled.

(a) $y = s\sin 60° = \frac{\sqrt{3}}{2}s$; $\frac{dy}{dt} = \frac{\sqrt{3}}{2}\frac{ds}{dt}$. When $\frac{ds}{dt} = 1000$, $\frac{dy}{dt} = \frac{\sqrt{3}}{2}(1000) = 500\sqrt{3}$ mi/h.

(b) $x = s\cos 60° = \frac{1}{2}s$; $\frac{dx}{dt} = \frac{1}{2}\frac{ds}{dt}$. When $\frac{ds}{dt} = 1000$, $\frac{dx}{dt} = \frac{1}{2}(1000) = 500\,\text{mi/h}$.

31. Differentiating $x = 1.25\tan\theta$ gives




$$\frac{dx}{dt} = 1.25\sec^2\theta\frac{d\theta}{dt}.$$

Converting 3° to $\frac{\pi}{60}$, we are given $\frac{d\theta}{dt} = \frac{\pi}{60}$. When $\theta = 60°$,

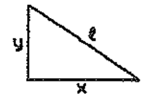
$$\frac{dx}{dt} = 1.25(\sec^2 60°)\left(\frac{\pi}{60}\right) = \frac{\pi}{12} \approx 0.26 \text{ km/s}.$$

32. The volume between the spheres is $V = \frac{4}{3}\pi r_o^3 - \frac{4}{3}\pi r_i^3$. Differentiating gives $\frac{dV}{dt} = 4\pi r_o^2\frac{dr_o}{dt} - 4\pi r_i^2\frac{dr_i}{dt}$. For $\frac{dr_o}{dt} = 2\,\text{m/hr}$, $\frac{dr_i}{dt} = -\frac{1}{2}\text{m/hr}$, $r_o = 3\,\text{m}$, and $r_i = 1\,\text{m}$, we have $\frac{dV}{dt} = 4\pi(9)(2) - 4\pi(1)(-1/2) = 74\pi\ \text{m}^3/\text{h}$.

33. The volume of a sphere is $V = \frac{4}{3}\pi r^3$. Differentiating gives $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$. The surface area of a sphere is $S = 4\pi r^2$, so $\frac{dV}{dt} = S\frac{dr}{dt}$. Since we are given that $\frac{dV}{dt} = kS$, we have $\frac{dr}{dt} = k$. Thus the radius changes at a constant rate.

34. The volume is $V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} = k$, where $k$ is a constant. Now, the surface area is $S = 4\pi r^2$ and $\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$. From the formula for $\frac{dV}{dt}$, we have $\frac{dr}{dt} = \frac{k}{4\pi r^2}$. Thus $\frac{dS}{dt} = 8\pi r\left(\frac{k}{4\pi r^2}\right) = \frac{2k}{r}$, and the rate of change of the surface area is inversely proportional to the radius.

**35.** From the Pythagorean Theorem, $x^2 + y^2 = \ell^2$. Differentiating gives

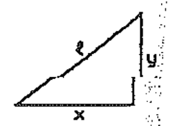



$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2\ell\frac{d\ell}{dt}.$$

We are given $\frac{dx}{dt} = 10$ knots and $\frac{dy}{dt} = 15$ knots, so $\frac{d\ell}{dt} = \frac{10x + 15y}{\ell}$. At 2:00 P.M., $x = 20$ nautical miles, $y = 15$ nautical miles, and $\ell = \sqrt{20^2 + 15^2} = 25$ nautical miles. Thus

$$\frac{d\ell}{dt} = \frac{10(20) + 15(15)}{25} = 17 \text{ knots}.$$

**36.** From the Pythagorean Theorem, $x^2 + y^2 = \ell^2$. Differentiating gives

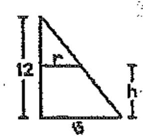



$$\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = \cancel{2}\ell\frac{d\ell}{dt}.$$

We are given $\frac{dy}{dt} = -9$ km/h and $\frac{dx}{dt} = 12$ km/h. Then $\frac{d\ell}{dt} = \frac{12x - 9y}{\ell}$. At 9:20 A.M., $y = 20 - \frac{4}{3}(9) = 8$ km, $x = \frac{4}{3}(12) = 16$ km, and $\ell = \sqrt{16^2 + 8^2} = 8\sqrt{5}$ km. Therefore

$$\frac{d\ell}{dt} = \frac{12(16) - 9(8)}{8\sqrt{5}} = 3\sqrt{5} \text{ km/h}.$$

**37.** Using similar triangles we observe $\frac{12 - h}{r} = \frac{12}{6}$ or $r = 6 - \frac{h}{2}$. The volume of the frustrum of the cone is

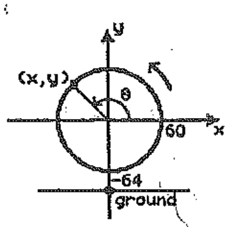



$$V = \frac{\pi h}{3}(r^2 + 6r + 36) = \frac{\pi h}{3}\left[\left(6 - \frac{h}{2}\right)^2 + 6\left(6 - \frac{h}{2}\right) + 36\right] = \frac{\pi}{12}h^3 - 3\pi h^2 + 36\pi h.$$

Then $\frac{dV}{dt} = \left(\frac{\pi}{4}h^2 - 6\pi h + 36\pi\right)\frac{dh}{dt}$, and for $\frac{dV}{dt} = 4\,\text{cm}^3/\text{s}$ we have

$$\frac{dh}{dt} = \frac{4}{\frac{\pi}{4}h^2 - 6\pi h + 36\pi} = \frac{16}{\pi(h - 12)^2} \text{ cm/s}.$$

**38.** Place the origin at the center of the ferris wheel with the $x$-axis parallel to the ground. To find the vertical and horizontal rates, we use $y = 60\sin\theta$ and $x = 60\cos\theta$. Differentiating we have $\frac{dy}{dt} = 60\cos\theta\frac{d\theta}{dt}$ and $\frac{dx}{dt} = -60\sin\theta\frac{d\theta}{dt}$. Assuming the wheel revolves counter-clockwise once




every two minutes, $\frac{d\theta}{dt} = \pi$ radians per minute. Thus, $\frac{dy}{dt} = 60\pi\cos\theta$ and $\frac{dx}{dt} = -60\pi\sin\theta$. When the passenger is 64 feet above the ground, $\theta = 0$, $\sin\theta = 0$, and $\cos\theta = 1$. Thus the passenger is rising $\frac{dy}{dt} = 60\pi$ ft/min, and is moving horizontally $\frac{dx}{dt} = 0$ ft/min.

**39.** Referring to the figure in Problem 38 we place the origin at the center of the wheel. The coordinates of the point $P$ are $x = 60\cos\theta$ and $y = 60\sin\theta$. If the coordinates of the point $Q$ are $(q, -64)$ then the slope of the line through $PQ$ is $\frac{60\sin\theta + 64}{60\cos\theta - q}$. Since this line is perpedicular to the line through the center of the wheel and $P$, its slope is also $-\frac{1}{\tan\theta}$. Solving $\frac{60\sin\theta + 64}{60\cos\theta - q} = -\frac{1}{\tan\theta}$ for $q$ we obtain $q = 60\cos\theta + 64\tan\theta + 60\sin\theta\tan\theta$ and $\frac{dq}{dt} = \left[-60\sin\theta + 64\sec^2\theta + 60\sin\theta\sec^2\theta + 60\sin\theta\right]\frac{d\theta}{dt}$. When $\theta = \pi/4$ and $d\theta/dt = \pi$ we have

$$\left.\frac{dq}{dt}\right|_{\theta=\pi/4} = (60\sqrt{2} + 128)\pi \approx 668.7 \text{ radians/min}.$$

40. (a) From $\frac{dP}{dt} = 800\frac{dm}{dt}$ and $\frac{dm}{dt} = 30$ kg/h we see that the momentum is changing at a rate of $800(30) = 24{,}000$ kg km/h.

(b) In this case both $m$ and $v$ are variables so $\frac{dP}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$. At $t = 1$ hour the mass of the airplane is $10^5 + 30 = 100{,}030$ kg and the velocity is 750 km/h. Thus

$$\left.\frac{dP}{dt}\right|_{t=1} = 100{,}030(20) + 750(30) = 2{,}023{,}100 \text{ kg km/h}^2.$$

41. (a) From $R = \frac{C}{T} = \frac{0.493T - 0.913}{T} = 0.493 - \frac{0.913}{T}$ we find $\frac{dR}{dt} = \frac{0.913}{T^2}\frac{dT}{dt} > 0$. Thus, the ratio increases.

(b) To find the value of $T$ when $C = \frac{T}{3}$ we solve $\frac{T}{3} = 0.493T - 0.913$, obtaining $T \approx 5.718$. Then

$$\left.\frac{dR}{dt}\right|_{T=5.718} \approx \frac{0.913}{(5.718)^2}(1) \approx 0.028 = 2.8\%/\text{day}.$$

42. The rate of change of length is $\frac{dL}{dt} = \frac{18-10}{20} = 0.4$ cm per million years. From $E = 0.007P^{2/3} = 0.007(0.12L^{2.53})^{2/3} \approx 0.0017L^{1.68667}$ we obtain $\frac{dE}{dt} \approx 0.0029L^{0.68667}\frac{dL}{dt} = 0.0029L^{0.68667}(0.4) = 0.00115L^{0.68667}$. To determine the value of $L$ when the fish was half its final body weight we note that the final body weight is $P = 0.12(18)^{2.53}$ and solve $\frac{1}{2}(0.12)(18)^{2.53} = 0.12L^{2.53}$. This gives $L \approx 13.69$ mm. Thus, the rate at which the specie's brain was growing is

$$\left.\frac{dE}{dt}\right|_{L=13.69} \approx 0.00115(13.69)^{0.68667} \approx 0.0069 \text{ g/million years.}$$