Unit 1 The Structure and Language of Math

I. The Complex Number System (1A, 1B)

Vocabulary:

ends — no end

* set - a finite or infinite group of elements
* element - an object in a set {usually numbers}
* subset - a portion of a set, set within a set

A. Specific sets of the Complex Number system

1. "$\mathbb{C}$" - the <u>complex</u> number set

   b can be 0

   * any number of the form $a+bi$
   * ex: $3+2i$, 5, -4, $\frac{2}{3}$, $\sqrt{13}$, π, $1.7923\ldots$

2. "$\mathbb{R}$" - the <u>real</u> number set
   * any number with no imaginary part
   * ex: 5, -4, $\frac{2}{3}$, $\sqrt{13}$, π, $1.7923\ldots$

3. "$\mathbb{Q}$" - the <u>rational</u> number set
   * can be written as a fraction, repeating decimal or terminating decimal
   * ex: 5, -4, $\frac{2}{3}$, $2.\overline{7}$, 3.25, $10.\overline{2378}$

4. "$\mathbb{Z}$" - the <u>integer</u> number set

   whole

   * positive and negative counting numbers
   * ex: 5, -4, 2, 0, 6, 1000000

5. "$\mathbb{N}$" - the <u>natural</u> number set
   * positive counting numbers
   * ex: 5, 2, 0, 6, 1000000

6. Irrational number set

* can't be written as a fraction, repeating decimal or terminating decimal

* ex: $\sqrt{2}$, π, e, 1.2793...

7. Imaginary number set

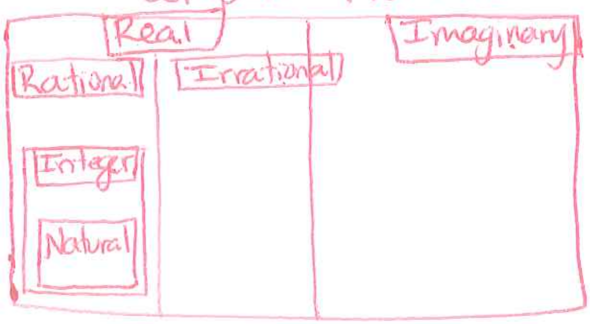
* numbers of the form $a+bi$, $b \neq 0$

* ex: $5i+1$, $-1-i$, $-10i$, $2+i$

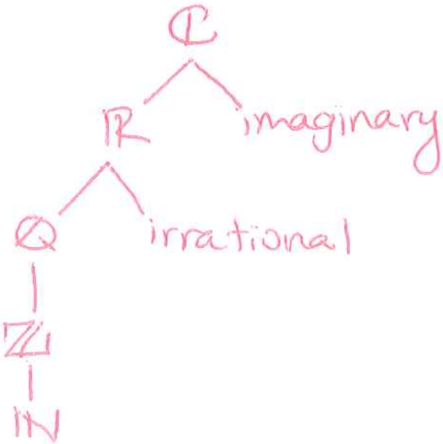
B. How Do the Number Sets Fit Together?

1. Diagram

Complex Numbers



2. Tree



C. Symbols Used With These Sets

1. "$\in$" - ... is an element of ...

* ex: $7 \in \mathbb{Z}$ means "7 is an element of the integer number set"

2. "$\subset$" - ... is a subset of ...

* ex: $\mathbb{N} \subset \mathbb{R}$ means "the natural number set is a subset of the real number set"

3. "$\perp\!\!\!\perp$" - ... is disjoint from ...

* ex: $\mathbb{Q} \perp\!\!\!\perp$ irrational means "The rational number set is disjoint from the irrational number set"

II. Set vs. Interval Notation (1C)

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Sets of numbers can be represented in 2 ways

A. Set Notation - uses brackets $\{\ \}$

1. A list of numbers

   ex: $\{4, 6, 8, 10\}$

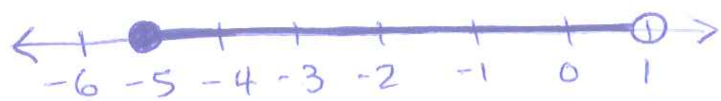
   visually:

2. An inequality

   ex: $\{x \mid -5 \leq x < 1\}$

   means: "x such that -5 is less than or equal to x is less than 1"

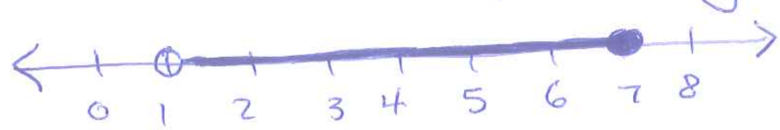
   visually:



B. Interval Notation - uses parenthesis & brackets ( ) & [ ] and represents <u>all</u> real numbers between two numbers.

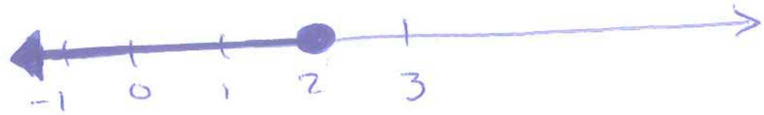
ex: $(1, 7]$ means: "all real numbers between 1 and 7, including 7."

visually:



ex: $(-\infty, 2]$

visually:



(or) is an open ○

[or] is a closed ●

C. The Intersection and Union of Sets

1. Symbols: "$\cap$" is the intersection of two sets, what they have in common.

   "$\cup$" is the union of two sets, what they have alltogether

   "$\varnothing$" is the null set or empty set

2. Examples:

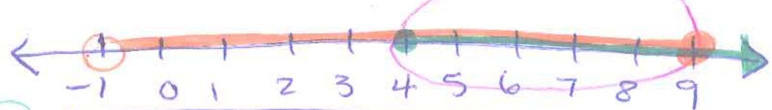
   A. Set Notation

   intersection

   intersection!

   ① $\{x \mid x \geq 4\} \cap \{x \mid -1 < x \leq 9\}$

   Draw:



   union

   $= \boxed{\{x \mid 4 \leq x \leq 9\}}$

   union!

   ② $\{x \mid x < 5\} \cup \{x \mid x \geq -7\}$

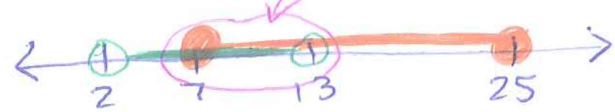
   Draw:

   $= \boxed{\mathbb{R}}$

   intersection

   B. Interval Notation

   ① $(2, 13) \cap [7, 25]$



   $= \boxed{[7, 13)}$

   union

   ② $[-5, 13) \cup (-7, 10)$

   $= \boxed{(-7, 13)}$

III. Closure (1E)

A set is <u>closed</u> under a certain operation $(+, -, \times, \div)$ if after applying the operation on any two numbers within the set, the result is also in the set.

$\boxed{\text{ex 1}}$ Is $\mathbb{N}$ closed under addition?

$$\underbrace{4}_{\mathbb{N}} + \underbrace{14}_{\mathbb{N}} = \underbrace{18}_{\mathbb{N}}$$

... and it happens <u>every</u> time

$\therefore$ yes, $\mathbb{N}$ is closed under addition.

$\boxed{\text{ex 2}}$ Is $\mathbb{Z}$ closed under division?

$$\underbrace{4}_{\mathbb{Z}} \div \underbrace{2}_{\mathbb{Z}} = \underbrace{2}_{\mathbb{Z}} \qquad \underbrace{7}_{\mathbb{Z}} \div \underbrace{3}_{\mathbb{Z}} = \underbrace{2.\overline{3}}_{\text{NOT } \mathbb{Z}}$$

{its $\mathbb{Q}$}

it only takes 1 counter-example to prove a statement is wrong!

$\therefore$ no, $\mathbb{Z}$ is not closed under division

IV Justification (1D)

Know <u>what</u> you're doing and <u>why</u> you're doing it.

A. Additive Inverse Property - adding the opp. to both sides

$$\begin{aligned} 3x + 8 &= 1 \\ -8 \quad & \; -8 \end{aligned}$$

B. Multiplicative Inverse Property - multiply by the reciprocal on both sides

$$\left(\tfrac{1}{\cancel{3}}\right)\cancel{3}x = -7\left(\tfrac{1}{3}\right)$$

C. Additive Identity - add 0, and the value stays the same

$$2 + 0 = 2$$

D. Multiplicative Identity - multiply by 1 and the value stays the same

$8 \cdot 1 = 8$

E. Reflexive Property - a number is equal to itself

$3 = 3$

F. Symmetric Property - "swith it"

if $a = b$ then $b = a$

G. Transitive -

if $a = b$ and $b = c$ then $a = c$

H. Zero Product Property - if the product of two numbers is 0, then one or both of the numbers must be 0.

if $a \cdot b = 0$ then $a = 0$, $b = 0$ or both

I. Distributive Property

$2(x+2) = 2x+4$ $\qquad$ $3a+6 = 3(a+2)$

$\boxed{\text{ex 1}}$ Solve and justify your steps: $x^2 - 5x + 6 = 0$

Process	Math	Justification
	$x^2 - 5x + 6 = 0$	
★ factor	$(x-2)(x-3) = 0$	★ Distributive Property
★ set both = 0	$x-2=0$ $\quad$ $x-3=0$ $+2 \ +2$ $\quad$ $+3 \ +3$	★ Zero Product Property
★ solve	$x = 2$ $\quad$ $x = 3$	★ Additive Inverse Property

what you're doing

I don't need this!

why you're doing it

I need this!

V Simplifying Expression (1F, 1G)

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A. Exponent / Radical Rules

1. $a^m \cdot a^n = a^{m+n}$
2. $(a^m)^n = a^{m \cdot n}$
3. $\frac{a^m}{a^n} = a^{m-n}$
4. $a^{-m} = \frac{1}{a^m}$
5. $a^0 = 1$
6. $a^{1/m} = \sqrt[m]{a}$

index

radicand

7. $a^{n/m} = \sqrt[m]{a^n} = (\sqrt[m]{a})^n$
8. $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$
9. $\sqrt{a} + \sqrt{a} = 2\sqrt{a}$

Be careful: $\sqrt{a} + \sqrt{b} = $:($= \sqrt{a} + \sqrt{b}$ can't add!

$\boxed{\text{ex 1}}$ Simplify the following expressions.

~~apply~~ the exponent

ⓐ $(2s^3t^{-1})^2 (\frac{1}{4}s^6)^{-2} (\frac{1}{8}t^4)$

$2^2 s^6 t^{-2} \cdot (\frac{1}{4})^{-2} s^{-12} \cdot \frac{1}{8} t^4$

$4 s^{-6} t^2 \cdot 16 \cdot \frac{1}{8} = \boxed{\frac{8t^2}{s^6}}$

ⓑ $\sqrt[4]{16a^5b^{-12}c^{15}} = 2ab^{-3}c^3\sqrt[4]{ac^3} = \boxed{\frac{2ac^3\sqrt[4]{ac^3}}{b^3}}$

ⓒ $\sqrt[3]{a^2b} \cdot \sqrt{a^4b} = (a^2b)^{1/3} \cdot (a^4b)^{1/2} = a^{2/3}b^{1/3} \cdot a^2b^{1/2}$

$a^{8/3} \cdot b^{5/6} = a^{16/6} \cdot b^{5/6} = \sqrt[6]{a^{16}b^5} = \boxed{a^2\sqrt[6]{a^4b^5}}$

ex2 Multiply

$$(\sqrt{3}+2\sqrt{7})(\sqrt{3}-\sqrt{5}) = \sqrt{\cancel{9}}^{\,3} - \sqrt{15} + 2\sqrt{21} - 2\sqrt{35}$$

$$\boxed{3 - \sqrt{15} + 2\sqrt{21} - 2\sqrt{35}}$$

ex3 Rationalize : justify your steps

multiplicative identity

ⓐ $$\frac{7}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{7\sqrt{3}}{2\cdot\sqrt{9}} = \boxed{\frac{7\sqrt{3}}{6}}$$

multiplicative identity

distributive property

ⓑ $$\frac{(x+2)}{\sqrt{x}+3} \cdot \frac{(\sqrt{x}-3)}{\sqrt{x}-3} = \frac{x\sqrt{x}-3x+2\sqrt{x}-6}{x-\cancel{3\sqrt{x}}+\cancel{3\sqrt{x}}-9}$$

conjugate

$$= \boxed{\frac{x\sqrt{x}-3x+2\sqrt{x}-6}{x-9}}$$

what's the difference?

multiplicative identity

distributive property

ⓒ $$\frac{(x+2)}{\sqrt{x+3}} \cdot \frac{\sqrt{x+3}}{\sqrt{x+3}} = \frac{x\sqrt{x+3}+2\sqrt{x+3}}{\sqrt{(x+3)^{2}}} =$$

$$\boxed{\frac{x\sqrt{x+3}+2\sqrt{x+3}}{x+3}}$$

C. Imaginary Numbers

repeats after 4!

$$\begin{cases} i = \sqrt{-1} \\ i^2 = -1 \\ i^3 = -i \\ i^4 = 1 \end{cases}$$

$i^5 = i$

ex1 Rationalize and justify with mathematical properties

multiplicative identity

ⓐ $\frac{2}{3i} \cdot \frac{i}{i} = \frac{2i}{3\cancel{i^2}_{-1}} = \boxed{\frac{2i}{-3}}$

multiplicative identity

distributive property

ⓑ $\frac{2}{3-i} \cdot \frac{3+i}{3+i} = \frac{6+2i}{9 - \cancel{i^2}_{+1}} = \frac{6+2i}{10} = \frac{3+i}{5} = \boxed{\frac{3}{5} + \frac{i}{5}}$

always break it into real + imag.

multiplicative identity

distributive property

ⓒ $\frac{1+i}{i-3} + \frac{7}{2i} = \left(\frac{i+3}{i+3}\right) \frac{1+i}{i-3} + \frac{7}{2i} \cdot \left(\frac{i}{i}\right) = \frac{i + \overset{-1}{\cancel{i^2}} + 3 + 3i}{\underset{-1}{\cancel{i^2}} - 9} + \frac{7i}{2\cancel{i^2}(-1)}$

mult. ID.

$= \frac{2+4i}{-10} + \frac{7i}{-2}\left(\frac{5}{5}\right) = \frac{2+4i}{-10} + \frac{35i}{-10} = \frac{2+39i}{-10}$

$= \frac{2}{-10} + \frac{39i}{-10}$

$\boxed{-\frac{1}{5} - \frac{39i}{10}}$

D. Simplifying Complex Fractions

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ex1 Simplify the following complex fractions and justify your steps with mathematical properties.

multiplicative identity

numerator

denominator

(a)
$$\frac{\frac{h+2}{\sqrt{h+1}}-2}{h} = \frac{\frac{h+2}{\sqrt{h+1}}-\frac{2\sqrt{h+1}}{\sqrt{h+1}}}{h} = \frac{\frac{h+2-2\sqrt{h+1}}{\sqrt{h+1}}}{h} =$$

★definition of division★ mult ID

$$\frac{h+2-2\sqrt{h+1}}{\sqrt{h+1}}\cdot\frac{1}{h} = \frac{h+2-2\sqrt{h+1}}{h\sqrt{h+1}}\cdot\frac{\sqrt{h+1}}{\sqrt{h+1}} =$$

distributive property

$$\boxed{\frac{h\sqrt{h+1}+2\sqrt{h+1}-2(h+1)}{h(h+1)}}$$

Goals:

★ no radicals in the denominator

★ one fraction

mult. ID

(b)
$$\sqrt{1+\left(\frac{x}{\sqrt{1-x^2}}\right)^2} = \sqrt{1+\frac{x^2}{1-x^2}} = \sqrt{\frac{1-x^2}{1-x^2}+\frac{x^2}{1-x^2}} =$$

mult. ID

$$\sqrt{\frac{1}{1-x^2}} = \frac{1}{\sqrt{1-x^2}}\cdot\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} = \boxed{\frac{\sqrt{1-x^2}}{1-x^2}}$$

mult ID dist. prop. definition of division

(c)
$$\frac{\frac{2i+1}{i+4}-3}{i} = \frac{\frac{2i+1}{i+4}-\frac{3(i+4)}{i+4}}{i} = \frac{\frac{2i+1-3i-12}{i+4}}{i} = \frac{-i-11}{i+4}\cdot\frac{1}{i}$$

mult ID dist. prop

$$\frac{-i-11}{\underset{-1}{\cancel{i^2}}+4i} = \frac{-i-11}{-1+4i}\cdot\frac{-1-4i}{-1-4i} = \frac{i+\overset{-4}{\cancel{4i^2}}+11+44i}{1-\underset{+16}{\cancel{16i^2}}} = \frac{7+45i}{17}$$

$$\boxed{\frac{7}{17}+\frac{45i}{17}}$$