Prep Session Topic: Particle Motion

Number Line for AB

Particle motion and similar problems are on the AP Calculus exams almost every year. The particle may be a "particle," a person, car, etc. The position, velocity or acceleration may be given as an equation, a graph or a table. There are a lot of different things students may be asked to find. While particles may not really move in this way, the question is a versatile way to test a variety of calculus concepts.

These questions may give the position equation, or the velocity equation or the acceleration equation along with an initial condition. Students may be asked about the motion of the particle: its direction, when it changes direction, its maximum position in one direction, etc. Speed, the absolute value of velocity, is also a common topic.

What you should know how to do:

- Move easily between the position, velocity and acceleration equations by differentiating or integrating.
- If you are given the velocity and an initial position, or given the acceleration and an initial velocity, you are looking at a differential equation initial value problem. Be sure you know how to do initial value problems.
- If you are given the velocity and an initial position, or given the acceleration and an initial velocity, you may often be able to approach the problem as an accumulation problems using either $x(t_1)=x(t_0)+\int_{t_0}^{t_1} v(t)\,dt$ or $v(t_1)=v(t_0)+\int_{t_0}^{t_1} a(t)\,dt$. This is often the easier that treating the situation as an initial value problem.
- Speed is the absolute value of velocity (it is not a vector quantity). <u>When the velocity and acceleration have the same sign the speed is increasing; if the signs are different the speed is decreasing</u>. Be sure you understand why this is true. This is a common topic on the AP exams.

  Graphically, speed is the non-directed distance from the velocity graph to the *t*-axis. If the distance of the velocity is increasing the speed is increasing. Reflecting the parts of the velocity graph that lie below the *t*-axis, will give you the graph of the speed.
- The total distance traveled at velocity $v(t)$ from $t = a$ to $t = b$ is given by $\int_a^b |v(t)|\,dt$. The net distance (displacement) over the same interval is $\int_a^b v(t)\,dt$.
- Don't be reluctant to use your graphing calculator for either of the computations in the previous bullet and to calculate $a(t)=v'(t)$.

Particle Motion

Contributed by Teresa Tarter, Bob Jones HS, Huntsville, AL

What you need to know about motion along the x-axis:

When you see…	**Think…**
Initially	$t = 0$
At rest	$v(t) = 0$
Particle moving right (forward or up)	$v(t) > 0$
Particle moving left (backward or down)	$v(t) < 0$
Average velocity on [a,b] [1]	$\frac{1}{b-a}\int_a^b v(t)dt = \frac{1}{b-a}x(t)\Big\vert_a^b = \frac{x(b)-x(a)}{b-a}$
Instantaneous velocity at time *t* = *a*	$v(a) = x'(a)$
Acceleration at time *t* = *c*	$a(c) = v'(c) = x''(c)$
Velocity is increasing	$a(t) = v'(a) > 0$
Velocity is decreasing	$a(t) = v'(a) < 0$
Speed	$\lvert v(t)\rvert$
Speed is increasing	$v(t)$ and $a(t)$ have same sign (both + or both -)
Speed is decreasing	$v(t)$ and $a(t)$ have different signs
Total distance traveled on [*a*, *b*]	$\int_a^b \lvert v(t)\rvert dt$ Absolute value is important!
Net distance traveled	$\int_a^b v(t)dt$
Position of object at time *t* = *b*	$x(b) = x(a) + \int_a^b v(t)dt$
Particle is farthest left (right)	Compare positions (*x*-values) at endpoints & at local minima (maxima).

[1] Do NOT think $\frac{v(a)+v(b)}{2}$

Student Notes

Free-response questions:

2002 AB3 Analytic stem

2008 AB4/BC4 Graph stem

2005 AB5 Graph stem

2006 AB 4 Table stem

Multiple-choice Selection: 2, 4, 5, 6, 7, 9, 10

No calculator: 1 – 6;

Graphing calculator allowed: 7 – 12.

Answers: 1 C, 2 B, 3 C, 4 D, 5 B, 6 E, 7 E, 8 B, 9 C, 10 D, 11 C, 12 B

WATCH and LISTEN to the multiple-choice questions being solved

Go to http://tinyurl.com/NMSI-Math-9 Click on the "Full Screen" arrow. Then click anywhere on the page to see and hear from that point on.

Click anywhere to go back anytime.

3. An object moves along the x-axis with initial position $x(0) = 2$. The velocity of the object at time $t \geq 0$ is given by $v(t) = \sin\left(\frac{\pi}{3}t\right)$.

   (a) What is the acceleration of the object at time $t = 4$?

   (b) Consider the following two statements.

   Statement I: For $3 < t < 4.5$, the velocity of the object is decreasing.
   Statement II: For $3 < t < 4.5$, the speed of the object is increasing.

   Are either or both of these statements correct? For each statement provide a reason why it is correct or not correct.

   (c) What is the total distance traveled by the object over the time interval $0 \leq t \leq 4$?

   (d) What is the position of the object at time $t = 4$?

END OF PART A OF SECTION II

Question 3

An object moves along the x-axis with initial position $x(0) = 2$. The velocity of the object at time $t \geq 0$ is given by $v(t) = \sin\left(\frac{\pi}{3}t\right)$.

(a) What is the acceleration of the object at time $t = 4$?

(b) Consider the following two statements.

Statement I: For $3 < t < 4.5$, the velocity of the object is decreasing.

Statement II: For $3 < t < 4.5$, the speed of the object is increasing.

Are either or both of these statements correct? For each statement provide a reason why it is correct or not correct.

(c) What is the total distance traveled by the object over the time interval $0 \leq t \leq 4$?

(d) What is the position of the object at time $t = 4$?

(a) $a(4) = v'(4) = \frac{\pi}{3}\cos\left(\frac{4\pi}{3}\right)$

$= -\frac{\pi}{6}$ or -0.523 or -0.524

1 : answer

(b) On $3 < t < 4.5$:

$a(t) = v'(t) = \frac{\pi}{3}\cos\left(\frac{\pi}{3}t\right) < 0$

Statement I is correct since $a(t) < 0$.

Statement II is correct since $v(t) < 0$ and $a(t) < 0$.

3 { 1 : I correct, with reason; 1 : II correct; 1 : reason for II }

(c) Distance $= \int_0^4 |v(t)|\,dt = 2.387$

OR

$x(t) = -\frac{3}{\pi}\cos\left(\frac{\pi}{3}t\right) + \frac{3}{\pi} + 2$

$x(0) = 2$

$x(4) = 2 + \frac{9}{2\pi} = 3.43239$

$v(t) = 0$ when $t = 3$

$x(3) = \frac{6}{\pi} + 2 = 3.90986$

$|x(3) - x(0)| + |x(4) - x(3)| = \frac{15}{2\pi} = 2.387$

3 {
1 : { limits of 0 and 4 on an integral of $v(t)$ or $|v(t)|$ or uses $x(0)$ and $x(4)$ to compute distance }
1 : handles change of direction at student's turning point
1 : answer
0/1 if incorrect turning point or no turning point
}

(d) $x(4) = x(0) + \int_0^4 v(t)\,dt = 3.432$

2 { 1 : integral; 1 : answer }

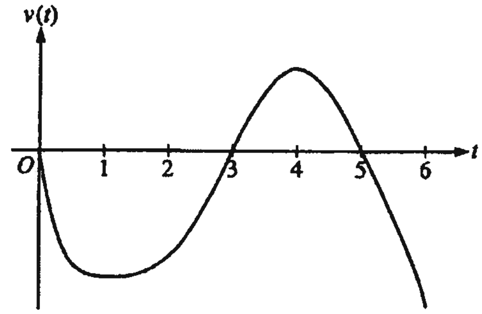
OR

$x(t) = -\frac{3}{\pi}\cos\left(\frac{\pi}{3}t\right) + \frac{3}{\pi} + 2$

$x(4) = 2 + \frac{9}{2\pi} = 3.432$

OR

2 { 1 : $x(t) = -\frac{3}{\pi}\cos\left(\frac{\pi}{3}t\right) + C$; 1 : answer; 0/1 if no constant of integration }



Graph of v

A particle moves along the x-axis so that its velocity at time t, for $0 \le t \le 6$, is given by a differentiable function v whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the t-axis and the graph of v on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

(a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

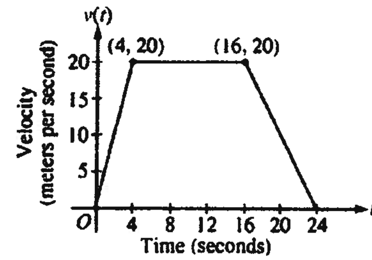
(b) For how many values of t, where $0 \le t \le 6$, is the particle at $x = -8$? Explain your reasoning.

(c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

(d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

(a) Since $v(t) < 0$ for $0 < t < 3$ and $5 < t < 6$, and $v(t) > 0$ for $3 < t < 5$, we consider $t = 3$ and $t = 6$. $x(3) = -2 + \int_0^3 v(t)\,dt = -2 - 8 = -10$ $x(6) = -2 + \int_0^6 v(t)\,dt = -2 - 8 + 3 - 2 = -9$ Therefore, the particle is farthest left at time $t = 3$ when its position is $x(3) = -10$.	$3 : \begin{cases} 1 : \text{identifies } t = 3 \text{ as a candidate} \\ 1 : \text{considers } \int_0^6 v(t)\,dt \\ 1 : \text{conclusion} \end{cases}$
(b) The particle moves continuously and monotonically from $x(0) = -2$ to $x(3) = -10$. Similarly, the particle moves continuously and monotonically from $x(3) = -10$ to $x(5) = -7$ and also from $x(5) = -7$ to $x(6) = -9$. By the Intermediate Value Theorem, there are 3 values of t for which the particle is at $x(t) = -8$.	$3 : \begin{cases} 1 : \text{positions at } t = 3,\ t = 5, \text{ and } t = 6 \\ 1 : \text{description of motion} \\ 1 : \text{conclusion} \end{cases}$
(c) The speed is decreasing on the interval $2 < t < 3$ since on this interval $v < 0$ and v is increasing.	1 : answer with reason
(d) The acceleration is negative on the intervals $0 < t < 1$ and $4 < t < 6$ since velocity is decreasing on these intervals.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

Question 5



A car is traveling on a straight road. For $0 \le t \le 24$ seconds, the car's velocity $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph above.

(a) Find $\int_0^{24} v(t)\,dt$. Using correct units, explain the meaning of $\int_0^{24} v(t)\,dt$.

(b) For each of $v'(4)$ and $v'(20)$, find the value or explain why it does not exist. Indicate units of measure.

(c) Let $a(t)$ be the car's acceleration at time t, in meters per second per second. For $0 < t < 24$, write a piecewise-defined function for $a(t)$.

(d) Find the average rate of change of v over the interval $8 \le t \le 20$. Does the Mean Value Theorem guarantee a value of c, for $8 < c < 20$, such that $v'(c)$ is equal to this average rate of change? Why or why not?

(a) $\int_0^{24} v(t)\,dt = \frac{1}{2}(4)(20) + (12)(20) + \frac{1}{2}(8)(20) = 360$ The car travels 360 meters in these 24 seconds.	$2 : \begin{cases} 1 : \text{value} \\ 1 : \text{meaning with units} \end{cases}$
(b) $v'(4)$ does not exist because $\lim_{t \to 4^-} \left(\frac{v(t) - v(4)}{t - 4} \right) = 5 \ne 0 = \lim_{t \to 4^+} \left(\frac{v(t) - v(4)}{t - 4} \right)$. $v'(20) = \frac{20 - 0}{16 - 24} = -\frac{5}{2}$ m/sec^2	$3 : \begin{cases} 1 : v'(4) \text{ does not exist, with explanation} \\ 1 : v'(20) \\ 1 : \text{units} \end{cases}$
(c) $a(t) = \begin{cases} 5 & \text{if } 0 < t < 4 \\ 0 & \text{if } 4 < t < 16 \\ -\frac{5}{2} & \text{if } 16 < t < 24 \end{cases}$ $a(t)$ does not exist at $t = 4$ and $t = 16$.	$2 : \begin{cases} 1 : \text{finds the values } 5, 0, -\frac{5}{2} \\ 1 : \text{identifies constants with correct intervals} \end{cases}$
(d) The average rate of change of v on $[8, 20]$ is $\frac{v(20) - v(8)}{20 - 8} = -\frac{5}{6}$ m/sec^2. No, the Mean Value Theorem does not apply to v on $[8, 20]$ because v is not differentiable at $t = 16$.	$2 : \begin{cases} 1 : \text{average rate of change of } v \text{ on } [8, 20] \\ 1 : \text{answer with explanation} \end{cases}$

Question 4

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

(a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

(c) Rocket B is launched upward with an acceleration of $a(t) = \dfrac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

(a) Average acceleration of rocket A is $\dfrac{v(80)-v(0)}{80-0} = \dfrac{49-5}{80} = \dfrac{11}{20}$ ft/sec^2	1 : answer
(b) Since the velocity is positive, $\int_{10}^{70} v(t)\,dt$ represents the distance, in feet, traveled by rocket A from $t = 10$ seconds to $t = 70$ seconds. A midpoint Riemann sum is $20[v(20)+v(40)+v(60)]$ $= 20[22+35+44] = 2020$ ft	3 : { 1 : explanation 1 : uses $v(20)$, $v(40)$, $v(60)$ 1 : value
(c) Let $v_B(t)$ be the velocity of rocket B at time t. $v_B(t) = \int \dfrac{3}{\sqrt{t+1}}\,dt = 6\sqrt{t+1} + C$ $2 = v_B(0) = 6 + C$ $v_B(t) = 6\sqrt{t+1} - 4$ $v_B(80) = 50 > 49 = v(80)$ Rocket B is traveling faster at time $t = 80$ seconds.	4 : { 1 : $6\sqrt{t+1}$ 1 : constant of integration 1 : uses initial condition 1 : finds $v_B(80)$, compares to $v(80)$, and draws a conclusion
Units of ft/sec^2 in (a) and ft in (b)	1 : units in (a) and (b)

Name: ________________________ **Class:** ___________________ **Date:** __________ **ID: A**

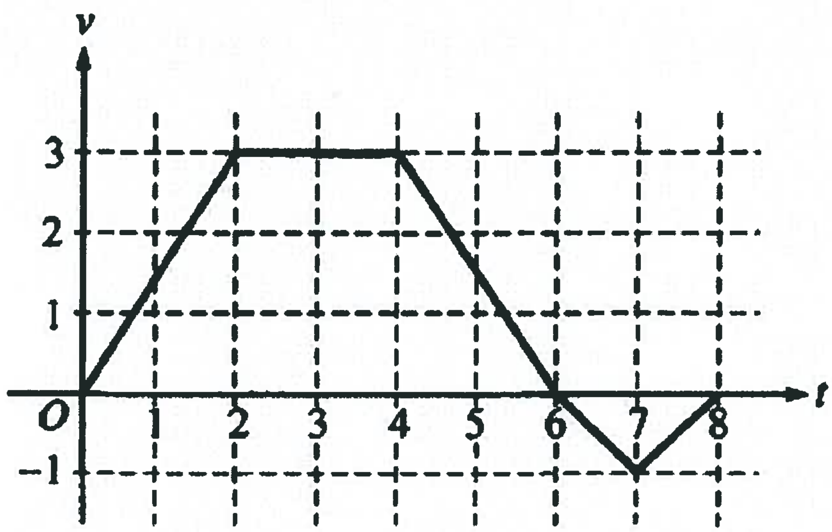
Particle Motion on a Line

Multiple Choice

Identify the choice that best completes the statement or answers the question.

Questions 1 - 6: No calculator allowed
Questions 7 - 12: Graphing calculator allowed

Questions 1 - 2 refer to the following situation.



A bug begins to crawl up a vertical wire at time $t = 0$. The velocity v of the bug at time t, $0 \le t \le 8$, is given by the function whose graph is shown above.

____ 1. At what value of t does the bug change direction?

a. 2
b. 4
c. 6
d. 7
e. 8

____ 2. What is the total distance the bug traveled from $t = 0$ to $t = 8$?

a. 14
b. 13
c. 11
d. 8
e. 6

____ 3. A particle moves along the x-axis so that its position at time t is given by $x(t) = t^2 - 6t + 5$. For what value of t is the velocity of the particle zero?

a. 1
b. 2
c. 3
d. 4
e. 5

____ 4. The maximum acceleration attained on the interval $0 \le t \le 3$ by the particle whose velocity is given by $v(t) = t^3 - 3t^2 + 12t + 4$ is

a. 9
b. 12
c. 14
d. 21
e. 40

____ 5. A particle moves along the x-axis so that its acceleration at any time t is $a(t) = 2t - 7$. If the initial velocity of the particle is 6, at what time t during the interval $0 \le t \le 4$ is the particle farthest to the right?

a. 0
b. 1
c. 2
d. 3
e. 4

____ 6. A particle moves along the *x-axis* so that at time $t \ge 0$ its position is given by $x(t) = 2t^3 - 21t^2 + 72t - 53$. At what time t is the particle at rest?

a. $t = 1$ only
b. $t = 3$ only
c. $t = \frac{7}{2}$ only
d. $t = 3$ and $t = \frac{7}{2}$
e. $t = 3$ and $t = 4$

t (sec)	0	2	4	6
$a(t)$ (ft/sec^2)	5	2	8	3

____ 7.

The data for the acceleration $a(t)$ of a car from 0 to 6 seconds are given in the table above. If the velocity at t =0 is 11 feet per second, the approximate value of the velocity at $t = 6$, computed using a left-hand Riemann sum with three subintervals of equal length, is

a. 26 ft/sec
b. 30 ft/sec
c. 37 ft/sec
d. 39 ft/sec
e. 41 ft/sec

____ 8. At time $t \geq 0$, the acceleration of a particle moving on the x-axis is $a(t) = t + \sin t$. At $t = 0$, the velocity of the particle is -2. For what value of t will the velocity of the particle be zero?

a. 1.02
b. 1.48
c. 1.85
d. 2.81
e. 3.14

____ 9. Aparticle moves along the x-axis so that at any time $t \geq 0$, its velocity is given by $v(t) = 3 + 4.1\cos(0.9t)$. What is the acceleration of the particle at time t =4?

a. -2.016
b. -0.677
c. 1.633
d. 1.814
e. 2.978

____ 10. The position of an object attached to a spring is given by $y(t) = \frac{1}{6}\cos(5t) - \frac{1}{4}\sin(5t)$, where t is time in seconds. In the first 4 seconds, how many times is the velocity of the object equal to 0?

a. Zero
b. Three
c. Five
d. Six
e. Seven

____ 11. A particle moves along the x-axis so that at any time $t \geq 0$, its velocity is given by $v(t) = \cos\left(2 - t^2\right)$. The position of the particle is 3 at time $t = 0$. What is the position of the particle when its velocity is first equal to 0?

a. 0.411
b. 1.310
c. 2.816
d. 3,091
e. 3.411

____ 12. The height h, in meters, of an object at time t is given by $h(t) = 24t + 24t^{\frac{3}{2}} - 16t^2$. What is the height of the object at the instant when it reaches its maximum upward velocity?

a. 2.545 meters
b. 10.263 meters
c. 34.125 meters
d. 54.889 meters
e. 89.005 meters