The Intermediate Value Theorem
Math 142, Section 01, Spring 2009

Intuitively, we think of a function $f(x)$ as being continuous at $x = a$ if the graph of $y = f(x)$ has no sudden breaks, jumps, or holes in it. Algebraically, this leads to requiring three conditions on $f(x)$:

(i.) $f(x)$ must be defined at $x = a$;
(ii.) $\lim_{x\to a} f(x)$ exists;
(iii.) $\lim_{x\to a} f(x) = f(a)$.

The first condition is concerned with what $f(x)$ does AT a. The second condition is concerned with what $f(x)$ NEAR a. The third condition can be interpreted as saying "$f(x)$ is continuous at a if the value of $f(x)$ AT a is consistent with the values of $f(x)$ NEAR a."

The Intermediate Value Theorem provides the formal justification of our intuition that a continuous function is one whose graph has no jumps or holes in it.

The Intermediate Value Theorem. *Suppose $f(x)$ is a continuous function defined on a closed interval $[a, b]$ and $f(a) \neq f(b)$. Then for any k between $f(a)$ and $f(b)$, there is (at least one) c in the open interval (a, b) satisfying $f(c) = k$.*

The *proof* of the Intermediate Value Theorem is out of our reach, as it relies on delicate properties of the real number system[1].

The *idea* of the Intermediate Value Theorem is not too difficult to grasp. It implies (among other things) that if a continuous function changes signs going from a to b, then the function had to have crossed the x-axis somewhere between a and b. The most difficult part is recognizing that the Intermediate Value Theorem can be used in a given problem.

Two things to keep in mind about the Intermediate Value Theorem. First, it only tells us there is a c so that $f(c) = k$. It does not tell us how many c's there are, nor does it tell us what the value of c is. Second, it is a statement about *continuous functions:* the conclusion of the theorem may or may not be satisfied for any particular discontinuous function. (Hopefully my interactive applets on discontinuous functions will be up and running on my course webpage by the time you read this.)

Example 1: Prove that there is a positive real number c satisfying $c^2 = 2$.

Solution: This problem is asking to prove the existence of the square root of 2. But isn't $\sqrt{2} = 1.414\ldots$? The problem lies in the . . . . For instance, 1.414 is NOT the square root of 2 since

$$1.414^2 = 1.999396 \neq 2.$$

At this point you might say "But you didn't use enough digits. My calculator says

$$\sqrt{2} = 1.41421356237$$

and upon squaring my calculator says

$$1.41421356237^2 = 2.00000000000,$$

so the square root of 2 exists an it is equal to 1.41421356237."

However, using Excel (formatting it to hold 30 digits) instead of a calculator, we find

$$1.41421356237^2 = 1.99999999999912500000000000000 \neq 2.$$

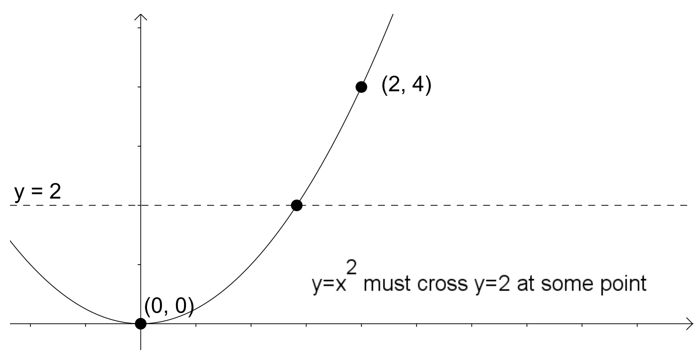
[1]A proof is usually given in courses like *Modern Analysis*, *Advanced Calculus*, or *Point-set Topology.*

Granted, this is extremely close to 2. It is so close to 2 that most calculators can't recognize the difference between them. Nevertheless, we have not found the actual square root of 2 as there still is a gap (albeit, a very small gap) between 1.99999999999125000000000000000 and 2.

This discussion *suggests* that we can find real numbers c so that c^2 is as close to 2 as we wish. How can we be sure this is true? Maybe something bizarre happens at very small scales? Maybe there is a fundamental level of precision that cannot be surpassed? Perhaps the best we can do is find numbers c with c^2 within $\pm 10^{-100}$ of 2, but no better[2]?

I hope this discussion convinces you that the phrase "The square root of two exists" is not as trivial as it first appears to be. Let's see how this relates to the Intermediate Value Theorem.

We want to show the existence of a real number c so that $c^2 = 2$. So let's consider the function $f(x) = x^2$. $f(x)$ is continuous everywhere, as it is a polynomial. We want to show that $f(c) = 2$ for some real number c, so by the Intermediate Value Theorem it is enough to show that at some point $f(x)$ is less that 2 and at some other point $f(x)$ is greater than 2. But $f(0) = 0^2 = 0$ and $f(2) = 2^2 = 4$. Applying the Intermediate Value Theorem to $f(x) = x^2$ on the interval $[0, 2]$ and taking $k = 2$, we are therefore guaranteed the existence of a number c satisfying $0 < c < 2$ so that $c^2 = 2$.



We have a bit of leeway in applying the Intermediate Value Theorem in situations like the above. In particular, rarely will we be given the endpoints a and b when we are supposed to apply the Intermediate Value Theorem. In the previous example I chose $a = 0$ and $b = 2$. Choosing $a = 1$ and $b = 2$ would have worked just as well, since $1^2 < 2 < 2^2$. For that matter, we could have picked $a = 1.3$ and $b = 19$ since $1.3^2 < 2 < 19^2$. We just needed to pick two specific numbers a and b so that $a^2 < 2 < b^2$.

Example 2: The polynomial $p(x) = 2x^3 + x^2 + x + 1$ has at least one real zero.
Try this one on your own before continuing. (A solution appears at the end of these notes):

More generally, the Intermediate Value Theorem can be used to show that any polynomial of odd degree has at least one real zero.

Here are some other applications.

[2]If you think this is far fetched, compare this with what happens in Quantum Mechanics. On macroscopic scales we can continually break physical matter up into smaller and smaller pieces. But the nature of matter fundamentally changes at the atomic and nuclear scales.

Example 3**:** There is a positive real number r satisfying $\cos r = r$.
Solution: This is an example of a *transcendental equation*. Essentially, this means that no matter how much mathematics you study, you will never be able to solve $\cos x = x$ algebraically. Nevertheless, the IVT guarantees there is a solution. To see this, let's consider the function $f(x) = \cos x - x$; notice this is continuous. We want to find a real number r satisfying $\cos r = r$, but this is the same as finding an r so that $f(r) = 0$. So, we just need to show that $f(x)$ is somewhere above 0 and somewhere below 0. Now,

$$f(0) = \cos 0 - 0 = 1$$

and

$$f(\frac{\pi}{2}) = \cos\frac{\pi}{2} - \frac{\pi}{2} = -\frac{\pi}{2}.$$

We picked 0 and $\pi/2$ since cosine is easy to evaluate at those two numbers. Thus, since $f(x)$ is continuous on $[0, \frac{\pi}{2}]$ and $f(0) > 0 > f(\frac{\pi}{2})$, the Intermediate Value Theorem guarantees the existence of a solution to $f(r) = 0$, and furthermore, there is a solution with $0 < r < \frac{\pi}{2}$.

This illustrates a useful trick: we often need to combine two functions into a single auxiliary function in order to apply the Intermediate Value Theorem.

Example 4**:** Show that $2^x = \frac{1}{x}$ has a real solution.
Try this one on your own before continuing.

The next example generalizes the last two. It says that if $f(x)$ starts off below $g(x)$, but at some later point $f(x)$ is above $g(x)$, then, assuming both functions are continuous, $f(x)$ and $g(x)$ must have crossed somewhere in between.

Example 5**:** Suppose $f(x)$ and $g(x)$ are each continuous on $[a, b]$. Suppose $f(a) < g(a)$ and $f(b) > g(b)$. Then there exists a $c \in (a, b)$ so that $f(c) = g(c)$.

Proof: Requiring $f(c) = g(c)$ is the same as requiring $f(c) - g(c) = 0$. This suggests looking at the function $F(x) = f(x) - g(x)$. $F(x)$ is continuous on $[a, b]$ since $f(x)$ and $g(x)$ are both continuous on $[a, b]$. Now,

$$F(a) = f(a) - g(a) < 0$$

since by assumption $f(a) < g(a)$. Also,

$$F(b) = f(b) - g(b) > 0$$

since by assumption $f(b) > g(b)$. The Intermediate Value Theorem therefore guarantees that $F(c) = 0$ for some $a < c < b$, and therefore (by the definition of $F(x)$) $f(c) = g(c)$.

Example 6**:** Suppose $g(x)$ is continuous on the interval $[0, 2]$ and $g(0) = g(2)$. Prove there must be a number c between 0 and 1 so that $g(x + 1) = g(x)$.

I'll let you try this one on your own.
HINTS:

(1) Looking at $f(x) = g(x + 1) - g(x)$ may be helpful.
(2) You may as well assume $g(1)$ is different from $g(0)$ and $g(2)$, since otherwise $x = 0$ satisfies the condition you are trying to prove.

This last example demonstrates the power of the Intermediate Value Theorem. We arrive at a nontrivial conclusion (that $g(c+1) = g(c)$ somewhere) while assuming rather little about the actual function ($g(x)$ is continuous and $g(0) = g(2)$.)

Solution to Example 2**:** We want to find a real number c so that $p(c) = 0$. $p(x)$ is a continuous everywhere since it is a polynomial. Thus, the Intermediate Value Theorem guarantees the existence of a real number c with $p(c) = 0$, provided we can show that $p(x)$ is less than zero somewhere and greater than zero somewhere else. Notice $p(0) = 1 > 0$. So we just need to find some place where $p(x) < 0$. But $p(-1) = 2(-1)^3 + (-1)^2 + (-1) + 1 = -1 < 0$. Thus, there is a real number c satisfying $p(c) = 0$. Moreover, we can find such a c between -1 and 0.

Solution to Example 4**:** We want to find a real number c so that $2^c = \frac{1}{c}$, which is the same as requiring $c2^c = 1$. We are therefore lead to the function $f(x) = x2^x$ which is continuous everywhere. Now, $f(0) = 0$ and $f(1) = 2$, so there is some c, $0 < c < 1$ for which $f(c) = 1$. Therefore, there is a $0 < c < 1$ satisfying $2^c = \frac{1}{c}$.

$f(x) = x2^x$ is not the only function you could have used. Did you use a different one?

Solution to Example 6**:** As the hint suggests, we look at $f(x) = g(x+1) - g(x)$. Also, as the hint suggests, if $g(0)$ happens to be equal to $g(1)$, then we simply take $c = 0$ and there is nothing left to prove. So we assume $g(0) \neq g(1)$. Now, the domain of $f(x)$ is $[0, 1]$. (Suppose you try to plug in $x = 1.5$, say. Then $f(1.5) = g(1.5 + 1) - g(1.5) = g(2.5) - g(1.5)$, but $g(2.5)$ is not necessarily defined. Thus, $f(x)$ is only defined on $[0, 1]$, not the full $[0, 2]$.) Also, being the difference of continuous functions, $f(x)$ is continuous. Now,

$$f(0) = g(1) - g(0)$$

and

$$f(1) = g(2) - g(1) = g(0) - g(1) = -(g(1) - g(0)) = -f(0).$$

Therefore, the continuous function $f(x)$ changes signs on the interval $[0, 1]$, so the Intermediate Value Theorem applies to show the existence of $0 < c < 1$ satisfying $f(c) = 0$. Finally, $0 = f(c) = g(c+1) - g(c)$ so $g(c+1) = g(c)$ for some c in $(0, 1)$.