Sections 5.1 & 5.2: Antiderivatives and Indefinite Integrals

Definition. A function F is called an **antiderivative** of f on an interval if $F'(x) = f(x)$ for all x in that interval.

Result. If F is an antiderivative of f on an interval, then the most general antiderivative of f on that interval is

$$F(x) + C,$$

where C is an arbitrary constant.

Definition. The set of all antiderivatives of the function f on an interval is called the **indefinite integral of** f and is denoted $\int f(x)\,dx$ (read "the integral of $f(x)$, dx"). Thus,

$$\int f(x)\,dx = F(x) + C,$$

where F is an antiderivative of f and C is an arbitrary constant called the **constant of integration.** The symbol "$\int$" is called the **integral sign** and $f(x)$ is called the **integrand**. The symbol "dx" has no official meaning by itself — its purpose is to show which variable the integral is with respect to. Note that *the integral sign must always be written with dx.*

Definition. The process of finding an antiderivative is called **integration**.

Example 1. Find $\displaystyle\int x^7\,dx$.

SOLUTION. An antiderivative of x^7 is $\frac{1}{8}x^8$. Hence, $\displaystyle\int x^7\,dx = \boxed{\tfrac{1}{8}x^8 + C}$.

Notice that we can check this result by differentiating:

$$F(x) = \tfrac{1}{8}x^8 + C$$

$$F'(x) = x^7$$

(The derivative of the constant C is just zero.)

When calculating an indefinite integral, it is very easy to check your answer. You simply take the derivative of your final result — it should be the original function (i.e., the integrand).

BASIC INTEGRATION FORMULAS

- $\int 1\,dx = x + C$
- $\int k\,dx = kx + C$, where k is any constant
- $\int x^n\,dx = \frac{1}{n+1}\,x^{n+1} + C$, for $n \neq -1$
- $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$
- $\int e^x\,dx = e^x + C$
- $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$, where k is any constant
- $\int kf(x)\,dx = k\int f(x)\,dx$, where k is any constant
- $\int \big[f(x) \pm g(x)\,dx\big] = \int f(x)\,dx \pm \int g(x)\,dx$

Example 2. Find $\int (4x^3 + 6x^2 - 7x + 3)\,dx$.

Example 3. Find $\displaystyle\int \left(3x^{1/2} + 2x^{2/3} + 4x^{-2} + 3x^{-1/2}\right)\, dx.$

Example 4. Find $\displaystyle\int \left(6\sqrt{x} - 5\sqrt[3]{x} + 2\sqrt[3]{x^2} - 3\right)\, dx.$

Example 5. Find $\displaystyle\int \left(\frac{4}{x} + \frac{5}{x^2} + \frac{6}{x^3} + \frac{7}{3\sqrt{x}}\right)\, dx.$

Example 6. Find $\displaystyle\int \left(\frac{3}{x} + e^{2x} + 5e^{-4x} - 7e^{3x}\right) dx.$

Example 7. Find $\displaystyle\int \frac{x^2 - 5x + 2}{x}\, dx.$

Example 8. Find $\displaystyle\int \frac{9x^3 + 8x^2 + 3x - 4}{3x^3}\, dx.$

EXERCISES

Find the following integrals. Check your answer by differentiating.

1. $\int (6x^2 - 4x + 3)\,dx$
2. $\int (4x^3 - 7x^2 + 3x - 5)\,dx$
3. $\int (2x^4 + 3x^3 - 5x^2 - 2x + 1)\,dx$
4. $\int (2x^4 + 6x^3 + 7x^2 - 4x + 1)\,dx$
5. $\int (5x^2 - 7x + 6)\,dx$
6. $\int (6x^3 - 6x^2 + 8x - 2)\,dx$
7. $\int (3x^{-2} + 4x^{-3} + 6)\,dx$
8. $\int (5x^{3/2} + 2x^{1/2} - 7x + 3)\,dx$
9. $\int (7x^{-3} + 3x^{-1/2} - 5x^{-3/2} - 6)\,dx$
10. $\int (3x^{-1} + 4e^{2x} + 5x - 6)\,dx$
11. $\int (7e^{3x} - 2x^{-1} + x + 3)\,dx$
12. $\int (2e^{-3x} - 4x^{-1} + 5)\,dx$
13. $\int (2e^{-x} + 3e^{x} + x^2 - 6)\,dx$
14. $\int \left(4\sqrt{x} + 3\sqrt[3]{x^2} - \frac{4}{x^2} - \frac{3}{x} + 2\right)\,dx$
15. $\int \left(\frac{5}{x^3} - \frac{2}{x} + \frac{3}{\sqrt{x}} + x + 1\right)\,dx$
16. $\int \left(\frac{1}{e^x} + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{\sqrt{x}} + 3\right)\,dx$
17. $\int \frac{x^2 + 5x + 1}{x}\,dx$
18. $\int \frac{3x^3 - 7x^2 + 4x - 1}{x}\,dx$
19. $\int \frac{2x^2 - 3x + 5}{x^2}\,dx$
20. $\int \frac{4x^3 - 8x^2 + 3x - 4}{2x^2}\,dx$

ANSWERS

1. $2x^3 - 2x^2 + 3x + C$
2. $x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 - 5x + C$
3. $\frac{2}{5}x^5 + \frac{3}{4}x^4 - \frac{5}{3}x^3 - x^2 + x + C$
4. $\frac{2}{5}x^5 + \frac{3}{2}x^4 + \frac{7}{3}x^3 - 2x^2 + x + C$
5. $\frac{5}{3}x^3 - \frac{7}{2}x^2 + 6x + C$
6. $\frac{3}{2}x^4 - 2x^3 + 4x^2 - 2x + C$
7. $-3x^{-1} - 2x^{-2} + 6x + C$
8. $2x^{5/2} + \frac{4}{3}x^{3/2} - \frac{7}{2}x^2 + 3x + C$
9. $-\frac{7}{2}x^{-2} + 6x^{1/2} + 10x^{-1/2} - 6x + C$
10. $3\ln|x| + 2e^{2x} + \frac{5}{2}x^2 - 6x + C$
11. $\frac{7}{3}e^{3x} - 2\ln|x| + \frac{1}{2}x^2 + 3x + C$
12. $-\frac{2}{3}e^{-3x} - 4\ln|x| + 5x + C$
13. $-2e^{-x} + 3e^{x} + \frac{1}{3}x^3 - 6x + C$
14. $\frac{8}{3}x^{3/2} + \frac{9}{5}x^{5/3} + 4x^{-1} - 3\ln|x| + 2x + C$
15. $-\frac{5}{2}x^{-2} - 2\ln|x| + 6x^{1/2} + \frac{1}{2}x^2 + x + C$
16. $-e^{-x} + \ln|x| - x^{-1} + 2x^{1/2} + 3x + C$
17. $\frac{1}{2}x^2 + 5x + \ln|x| + C$
18. $x^3 - \frac{7}{2}x^2 + 4x - \ln|x| + C$
19. $2x - 3\ln|x| - 5x^{-1} + C$
20. $x^2 - 4x + \frac{3}{2}\ln|x| + 2x^{-1} + C$