Mathematics Learning Centre

The University of Sydney

Derivatives of exponential and logarithmic functions

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1 Derivatives of exponential and logarithmic functions

If you are not familiar with exponential and logarithmic functions you may wish to consult the booklet *Exponents and Logarithms* which is available from the Mathematics Learning Centre.

You may have seen that there are two notations popularly used for natural logarithms, $\log_e$ and ln. These are just two different ways of writing exactly the same thing, so that $\log_e x \equiv \ln x$. In this booklet we will use both these notations.

The basic results are:

$$\begin{aligned} \frac{d}{dx}e^x &= e^x \\ \frac{d}{dx}(\log_e x) &= \frac{1}{x}. \end{aligned}$$

We can use these results and the rules that we have learnt already to differentiate functions which involve exponentials or logarithms.

Example

Differentiate $\log_e(x^2 + 3x + 1)$.

Solution

We solve this by using the chain rule and our knowledge of the derivative of $\log_e x$.

$$\begin{aligned} \frac{d}{dx}\log_e(x^2+3x+1) &= \frac{d}{dx}(\log_e u) \qquad \text{(where } u = x^2+3x+1\text{)} \\ &= \frac{d}{du}(\log_e u) \times \frac{du}{dx} \qquad \text{(by the chain rule)} \\ &= \frac{1}{u} \times \frac{du}{dx} \\ &= \frac{1}{x^2+3x+1} \times \frac{d}{dx}(x^2+3x+1) \\ &= \frac{1}{x^2+3x+1} \times (2x+3) \\ &= \frac{2x+3}{x^2+3x+1}. \end{aligned}$$

Example

Find $\frac{d}{dx}(e^{3x^2})$.

Solution

This is an application of the chain rule together with our knowledge of the derivative of e^x.

$$\begin{aligned}
\frac{d}{dx}(e^{3x^2}) &= \frac{de^u}{dx} \qquad \text{where } u = 3x^2 \\
&= \frac{de^u}{du} \times \frac{du}{dx} \qquad \text{by the chain rule} \\
&= e^u \times \frac{du}{dx} \\
&= e^{3x^2} \times \frac{d}{dx}(3x^2) \\
&= 6xe^{3x^2}.
\end{aligned}$$

Example

Find $\frac{d}{dx}(e^{x^3+2x})$.

Solution

Again, we use our knowledge of the derivative of e^x together with the chain rule.

$$\begin{aligned}
\frac{d}{dx}(e^{x^3+2x}) &= \frac{de^u}{dx} \qquad (\text{where } u = x^3 + 2x) \\
&= e^u \times \frac{du}{dx} \qquad (\text{by the chain rule}) \\
&= e^{x^3+2x} \times \frac{d}{dx}(x^3 + 2x) \\
&= (3x^2 + 2) \times e^{x^3+2x}.
\end{aligned}$$

Example

Differentiate $\ln(2x^3 + 5x^2 - 3)$.

Solution

We solve this by using the chain rule and our knowledge of the derivative of $\ln x$.

$$\begin{aligned}
\frac{d}{dx}\ln(2x^3 + 5x^2 - 3) &= \frac{d\ln u}{dx} \qquad (\text{where } u = (2x^3 + 5x^2 - 3) \\
&= \frac{d\ln u}{du} \times \frac{du}{dx} \qquad (\text{by the chain rule}) \\
&= \frac{1}{u} \times \frac{du}{dx} \\
&= \frac{1}{2x^3 + 5x^2 - 3} \times \frac{d}{dx}(2x^3 + 5x^2 - 3) \\
&= \frac{1}{2x^3 + 5x^2 - 3} \times (6x^2 + 10x) \\
&= \frac{6x^2 + 10x}{2x^3 + 5x^2 - 3}.
\end{aligned}$$

There are two shortcuts to differentiating functions involving exponents and logarithms. The four examples above gave

$$\begin{aligned}
\frac{d}{dx}(\log_e(x^2+3x+1)) &= \frac{2x+3}{x^2+3x+1} \\
\frac{d}{dx}(e^{3x^2}) &= 6xe^{3x^2} \\
\frac{d}{dx}(e^{x^3+2x}) &= (3x^2+2)e^{3x^2} \\
\frac{d}{dx}(\log_e(2x^3+5x^2-3)) &= \frac{6x^2+10x}{2x^3+5x^2-3}.
\end{aligned}$$

These examples suggest the general rules

$$\begin{aligned}
\frac{d}{dx}(e^{f(x)}) &= f'(x)e^{f(x)} \\
\frac{d}{dx}(\ln f(x)) &= \frac{f'(x)}{f(x)}.
\end{aligned}$$

These rules arise from the chain rule and the fact that $\frac{de^x}{dx} = e^x$ and $\frac{d\ln x}{dx} = \frac{1}{x}$. They can speed up the process of differentiation but it is not necessary that you remember them. If you forget, just use the chain rule as in the examples above.

Exercise 1

Differentiate the following functions.

a. $f(x) = \ln(2x^3)$ **b.** $f(x) = e^{x^7}$ **c.** $f(x) = \ln(11x^7)$

d. $f(x) = e^{x^2+x^3}$ **e.** $f(x) = \log_e(7x^{-2})$ **f.** $f(x) = e^{-x}$

g. $f(x) = \ln(e^x + x^3)$ **h.** $f(x) = \ln(e^x x^3)$ **i.** $f(x) = \ln\left(\frac{x^2+1}{x^3-x}\right)$

Solutions to Exercise 1

a. $f'(x) = \dfrac{6x^2}{2x^3} = \dfrac{3}{x}$

Alternatively write $f(x) = \ln 2 + 3\ln x$ so that $f'(x) = 3\dfrac{1}{x}$.

b. $f'(x) = 7x^6e^{x^7}$

c. $f'(x) = \frac{7}{x}$

d. $f'(x) = (2x + 3x^2)e^{x^2+x^3}$

e. Write $f(x) = \log_e 7 - 2\log_e x$ so that $f'(x) = -\frac{2}{x}$.

f. $f'(x) = -e^{-x}$

g. $f'(x) = \dfrac{e^x + 3x^2}{e^x + x^3}$

h. Write $f(x) = \ln e^x + \frac{3}{\ln x}$ so that $f'(x) = 1 + \dfrac{3}{x}$.

i. Write $f(x) = \ln(x^2 + 1) - \ln(x^3 - x)$ so that $f'(x) = \dfrac{2x}{x^2+1} - \dfrac{3x^2-1}{x^3-x}$.