Derivatives of exponential and logarithmic functions

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1 Derivatives of exponential and logarithmic functions

If you are not familiar with exponential and logarithmic functions you may wish to consult the booklet *Exponents and Logarithms* which is available from the Mathematics Learning Centre.

You may have seen that there are two notations popularly used for natural logarithms, \( \log_e x \) and \( \ln x \). These are just two different ways of writing exactly the same thing, so that \( \log_e x \equiv \ln x \). In this booklet we will use both these notations.

The basic results are:

\[
\frac{d}{dx} e^x = e^x \\
\frac{d}{dx} (\log_e x) = \frac{1}{x}.
\]

We can use these results and the rules that we have learnt already to differentiate functions which involve exponentials or logarithms.

**Example**

Differentiate \( \log_e (x^2 + 3x + 1) \).

**Solution**

We solve this by using the chain rule and our knowledge of the derivative of \( \log_e x \).

\[
\frac{d}{dx} \log_e (x^2 + 3x + 1) = \frac{d}{dx} (\log_e u) \quad \text{(where } u = x^2 + 3x + 1) \\
= \frac{d}{du} (\log_e u) \times \frac{du}{dx} \quad \text{(by the chain rule)} \\
= \frac{1}{u} \times \frac{du}{dx} \\
= \frac{1}{x^2 + 3x + 1} \times \frac{d}{dx} (x^2 + 3x + 1) \\
= \frac{1}{x^2 + 3x + 1} \times (2x + 3) \\
= \frac{2x + 3}{x^2 + 3x + 1}.
\]

**Example**

Find \( \frac{d}{dx}(e^{3x^2}) \).
Solution
This is an application of the chain rule together with our knowledge of the derivative of $e^x$.

\[
\frac{d}{dx}(e^{3x^2}) = \frac{de^u}{dx} \quad \text{where } u = 3x^2
\]
\[
= \frac{de^u}{du} \times \frac{du}{dx} \quad \text{by the chain rule}
\]
\[
= e^u \times \frac{du}{dx}
\]
\[
= e^{3x^2} \times \frac{d}{dx}(3x^2)
\]
\[
= 6xe^{3x^2}.
\]

Example
Find $\frac{d}{dx}(e^{x^3+2x})$.

Solution
Again, we use our knowledge of the derivative of $e^x$ together with the chain rule.

\[
\frac{d}{dx}(e^{x^3+2x}) = \frac{de^u}{dx} \quad \text{(where } u = x^3 + 2x)\]
\[
= e^u \times \frac{du}{dx} \quad \text{(by the chain rule)}
\]
\[
= e^{x^3+2x} \times \frac{d}{dx}(x^3 + 2x)
\]
\[
= (3x^2 + 2) \times e^{x^3+2x}.
\]

Example
Differentiate $\ln (2x^3 + 5x^2 - 3)$.

Solution
We solve this by using the chain rule and our knowledge of the derivative of $\ln x$.

\[
\frac{d}{dx} \ln (2x^3 + 5x^2 - 3) = \frac{d \ln u}{dx} \quad \text{(where } u = (2x^3 + 5x^2 - 3)\}
\]
\[
= \frac{d \ln u}{du} \times \frac{du}{dx} \quad \text{(by the chain rule)}
\]
\[
= \frac{1}{u} \times \frac{du}{dx}
\]
\[
= \frac{1}{2x^3 + 5x^2 - 3} \times \frac{d}{dx}(2x^3 + 5x^2 - 3)
\]
\[
= \frac{1}{2x^3 + 5x^2 - 3} \times (6x^2 + 10x)
\]
\[
= \frac{6x^2 + 10x}{2x^3 + 5x^2 - 3}.
\]
There are two shortcuts to differentiating functions involving exponents and logarithms. The four examples above gave

\[
\frac{d}{dx} (\log_e(x^2 + 3x + 1)) = \frac{2x + 3}{x^2 + 3x + 1}
\]
\[
\frac{d}{dx} (e^{3x^2}) = 6xe^{3x^2}
\]
\[
\frac{d}{dx} (e^{x^3+2x}) = (3x^2 + 2)e^{3x^2}
\]
\[
\frac{d}{dx} (\log_e(2x^3 + 5x^2 - 3)) = \frac{6x^2 + 10x}{2x^3 + 5x^2 - 3}.
\]

These examples suggest the general rules

\[
\frac{d}{dx} (e^{f(x)}) = f'(x)e^{f(x)}
\]
\[
\frac{d}{dx} (\ln f(x)) = \frac{f'(x)}{f(x)}.
\]

These rules arise from the chain rule and the fact that \( \frac{dx}{dx} = e^x \) and \( \frac{d}{dx} \ln x = \frac{1}{x} \). They can speed up the process of differentiation but it is not necessary that you remember them. If you forget, just use the chain rule as in the examples above.

**Exercise 1**

Differentiate the following functions.

a. \( f(x) = \ln(2x^3) \)  
   b. \( f(x) = e^{x^7} \)  
   c. \( f(x) = \ln(11x^7) \)

d. \( f(x) = e^{x^2+x^3} \)  
   e. \( f(x) = \log_e(7x^{-2}) \)  
   f. \( f(x) = e^{-x} \)

g. \( f(x) = \ln(e^x + x^3) \)  
   h. \( f(x) = \ln(e^{x^3}) \)  
   i. \( f(x) = \ln \left( \frac{x^2 + 1}{x^3 - x} \right) \)
Solutions to Exercise 1

a. \( f'(x) = \frac{6x^2}{2x^3} = \frac{3}{x} \)

Alternatively write \( f(x) = \ln 2 + 3 \ln x \) so that \( f'(x) = 3 \frac{1}{x} \).

b. \( f'(x) = 7x^6e^x \)

c. \( f'(x) = \frac{7}{x} \)

d. \( f'(x) = (2x + 3x^2)e^{x^2+x^3} \)

e. Write \( f(x) = \log_e 7 - 2 \log_e x \) so that \( f'(x) = -\frac{2}{x} \).

f. \( f'(x) = -e^{-x} \)

g. \( f'(x) = \frac{e^x + 3x^2}{e^x + x^3} \)

h. Write \( f(x) = \ln e^x + \frac{3}{\ln x} \) so that \( f'(x) = 1 + \frac{3}{x} \).

i. Write \( f(x) = \ln(x^2 + 1) - \ln(x^3 - x) \) so that \( f'(x) = \frac{2x}{x^2 + 1} - \frac{3x^2 - 1}{x^3 - x} \).