AP Calculus — Miss Brown

Differential Equation Applications — January 6, 2011

1. $\frac{dR}{dt} = KR$

2. $\frac{dU}{dt} = KU^2$

3. $\frac{dT}{dt} = K(T - T_e)$ where $T > T_e$

4. $\frac{dB}{dt} = KB$

$\frac{dB}{B} = K\,dt$

$\int \frac{dB}{B} = \int K\,dt$

$\ln|B| + C_1 = Kt + C_2$

$\ln|B| = Kt + C$

$e^{\ln|B|} = e^{Kt + C}$

$B = e^{Kt} \cdot e^{C}$

$B = Ce^{Kt}$

Find value of C: $t = 0$, $B = 1000$

$1000 = Ce^{K(0)}$

$1000 = Ce^{0}$

$C = 1000$ $\therefore$ $B = 1000e^{Kt}$

Find value of K: $t = 10$ $B = 5000$

$5000 = 1000e^{K(10)}$ $K = \frac{\ln 5}{10}$

$5 = e^{10K}$

$\ln 5 = \ln e^{10K}$

$\ln 5 = 10K$ $\therefore$ $B = 1000e^{\frac{\ln 5}{10}t}$

Find B when $t = 15$

$$B = 1000e^{\frac{\ln 5}{10}t}$$

$$B = 1000e^{\frac{\ln 5}{10}(15)}$$ ← use calculator to evaluate

$$B = 11180.340$$

At $t = 15$, there are 11180 whole bacteria.

5. $\frac{dR}{dt} = KR$

$\frac{dR}{R} = K\,dt$

$\int \frac{dR}{R} = \int K\,dt$

$\ln|R| + C_1 = Kt + C_2$

$\ln|R| = Kt + C$

$e^{\ln|R|} = e^{Kt+C}$

$R = e^{Kt} \cdot e^{C}$

$R = Ce^{Kt}$

Find value of C: $t=0,\ R=200$

$200 = Ce^{K(0)}$

$200 = Ce^{0}$

$C = 200 \qquad \therefore\ R = 200e^{Kt}$

Find value of K: $t = 100,\ R = 180$

$180 = 200e^{K(100)}$

$\frac{9}{10} = e^{100K}$

$\ln\frac{9}{10} = \cancel{\ln} e^{100K}$

$\ln\frac{9}{10} = 100K$

$K = \frac{\ln\frac{9}{10}}{100} \qquad \therefore\ R = 200e^{\frac{\ln\frac{9}{10}}{100}t}$

Find R when $t = 500$

$R = 200e^{\frac{\ln\frac{9}{10}}{100}(500)}$ (Use calculator to evaluate)

$R = 118.098$

After 500 years, there will be 118.098 mg remaining.

Half-Life → How long will it take to reduce by half?

$t=0$, $R=200$
$t=?$, $R=100$ ← half of original amount

$R = 200e^{\frac{\ln\frac{9}{10}}{100}t}$

$100 = 200e^{\frac{\ln\frac{9}{10}}{100}t}$

$\frac{1}{2} = e^{\frac{\ln\frac{9}{10}}{100}t}$

$\ln\frac{1}{2} = \ln e^{\frac{\ln\frac{9}{10}}{100}t}$

$\ln\frac{1}{2} = \frac{\ln\frac{9}{10}}{100}t$

$100\ln\frac{1}{2} = \ln\frac{9}{10}t$

$t = \frac{100\ln\frac{1}{2}}{\ln\frac{9}{10}}$ ← use calculator to evaluate.

$t = 657.881$

The half-life of radium is 657.881 years.

6. $\frac{dB}{dt} = kB$

$\frac{dB}{B} = k\,dt$

$\int \frac{dB}{B} = \int k\,dt$

$\ln|B| + C_1 = kt + C_2$

$\ln|B| = kt + C$

$e^{\ln|B|} = e^{kt+C}$

$B = e^{kt} \cdot e^{C}$

$B = Ce^{kt}$

Find value of C : $t=0$, $B=1$

The problem doesn't tell us the original amount. Since it tells us it triples in 8 hours, I can pick any # for the original and just triple it for $t=8$. So, $t=0$, $B=1$ and $t=8$, $B=3$. If you choose a different original, you will get the same answers in the end.

$1 = Ce^{k(0)}$

$1 = Ce^{0}$

$C=1 \quad \therefore B = e^{kt}$

Find value of k : $t=8$, $B=3$

$3 = e^{k(1)}$

$3 = e^{k}$

$\ln 3 = \ln e^{k}$

$\ln 3 = k \quad \therefore B = e^{(\ln 3)t}$

Find B when $t=12$

$B = e^{(\ln 3)(12)}$ (Use calculator to evaluate)

$B = 531441$

There at 531441 bacteria after 12 hours.

7. $\frac{dP}{dt} = k(800-P)$

$$\frac{dP}{800-P} = k\,dt$$

$$\int \frac{dP}{800-P} = \int k\,dt$$

$$\left(\begin{array}{l|l} \int \frac{dP}{800-P} & \\ -\int \frac{du}{u} & u = 800-P \\ -\ln|u|+C & \frac{du}{dP} = -1 \\ -\ln|800-P|+C & -du = dP \end{array} \right)$$

$$-\ln|800-P| + C_1 = kt + C_2$$

$$-\ln|800-P| = kt + C$$

$$\ln|800-P| = -kt + C \qquad \left(C = \frac{C}{-1}\right)$$

$$e^{\ln|800-P|} = e^{-kt+C}$$

$$800-P = e^{-kt} \cdot e^{C}$$

$$800-P = Ce^{-kt}$$

$$P = 800 - Ce^{-kt}$$

Find the value of C: $\quad P(0) = 500$

$$\underset{\uparrow t}{P(0)} = \underset{\uparrow P}{500}$$

$$500 = 800 - Ce^{-k(0)}$$

$$500 = 800 - Ce^{0}$$

$$500 = 800 - C$$

$$-300 = -C$$

$$C = 300 \qquad \therefore \qquad P = 800 - 300e^{-kt}$$

A. $P = 800 - 300e^{-kt}$

B. Find the value of k: $\quad P(2) = 700$

$$\underset{\uparrow t}{P(2)} = \underset{\uparrow P}{700}$$

$$700 = 800 - 300e^{-k(2)}$$

$$700 = 800 - 300e^{-2k}$$

$$-100 = -300e^{-2k}$$

$$\frac{1}{3} = e^{-2k}$$

$$\ln\frac{1}{3} = \ln e^{-2k}$$

$$\ln\frac{1}{3} = -2k$$

$$k = \frac{\ln\frac{1}{3}}{-2}$$

C. $P = 800 - 300e^{-\frac{\ln\frac{1}{3}}{-2}t} = 800 - 300e^{\frac{\ln\frac{1}{3}}{2}t}$

If we break down the 'k' value, we get...

$\frac{\ln\frac{1}{3}}{2} = \frac{\ln 1 - \ln 3}{2} = \frac{0 - \ln 3}{2} = \frac{-\ln 3}{2}$

$\lim_{t \to \infty} 800 - 300e^{-\frac{\ln 3}{2}t}$

$\lim_{t \to \infty} 800 - \frac{300}{e^{\frac{\ln 3}{2}t}}$ ② This term will therefore go to zero at t grows toward ∞.

① The denominator will grow toward ∞ as t grows toward ∞.

800

8. $\frac{dF}{dt} = -0.1(F-70)$

$\frac{dF}{F-70} = -0.1\,dt$

$\int \frac{dF}{F-70} = \int -0.1\,dt$

$\ln|F-70| + C_1 = -0.1t + C_2$

$\ln|F-70| = -0.1t + C$

$e^{\ln|F-70|} = e^{-0.1t + C}$

$F-70 = e^{-0.1t} \cdot e^{C}$

$F-70 = Ce^{-0.1t}$

$F = 70 + Ce^{-0.1t}$

Find the value of C: $F(0) = 180$

$180 = 70 + Ce^{-0.1(0)}$

$180 = 70 + Ce^{0}$

$180 = 70 + C$

$C = 110$ $\therefore$ $F = 70 + 110e^{-0.1t}$

A. $F = 70 + 110e^{-0.1t}$

B. Find F when $t = 10$.

$F = 70 + 110e^{-0.1(10)}$ (use calculator to evaluate)

$F = 110.467°F$

After 10 minutes, the tea is 110.467° F.

c. Find t when $F = 120$.

$$120 = 70 + 110e^{-0.1t}$$

$$50 = 110e^{-0.1t}$$

$$\frac{5}{11} = e^{-0.1t}$$

$$\ln\frac{5}{11} = \ln e^{-0.1t}$$

$$\ln\frac{5}{11} = -0.1t$$

$$t = \frac{\ln\frac{5}{11}}{-0.1}$$

$$t = 7.885$$

The tea will be safe to drink after 8 minutes.

9. $\frac{dQ}{dt} = K(450 - Q)$

$\frac{dQ}{450-Q} = K\,dt$

$\int \frac{dQ}{450-Q} = \int K\,dt$

$-\ln|450-Q| + C_1 = Kt + C_2$

$-\ln|450-Q| = Kt + C$

$\ln|450-Q| = Kt + C$ (where $C = \frac{C}{-1}$ + $K = \frac{K}{-1}$)

$e^{\ln|450-Q|} = e^{Kt+C}$

$450 - Q = e^{Kt} \cdot e^{C}$

$450 - Q = Ce^{Kt}$

$Q = 450 - Ce^{Kt}$

Find the value of C: $t=0$, $Q = 100$

$100 = 450 - Ce^{K(0)}$

$-350 = -Ce^{0}$

$C = +350$ $\therefore$ $Q = 450 - 350e^{Kt}$

Find the value of K: $t = 2$, $Q = 300$

$300 = 450 - 350e^{K(2)}$

$-150 = -350e^{2K}$

$\frac{3}{7} = e^{2K}$

$\ln\frac{3}{7} = \ln e^{2K}$

$\ln\frac{3}{7} = 2K$

$K = \frac{\ln\frac{3}{7}}{2}$ $\therefore$ $Q = 450 - 350e$

$Q = 450 - 350e$

A. $Q = 450 - 350e^{\frac{\ln\frac{3}{7}}{2}t}$ Find Q when $t = 3$.

$Q = 450 - 350e^{\frac{\ln\frac{3}{7}}{2}(3)}$ (Use calculator to evaluate.)

$Q = 351.802$

In 1993, the population was 351,802 jellyfish.

B. Find t when $Q = 400$

$Q = 450 - 350e^{\frac{\ln\frac{3}{7}}{2}t}$

$400 = 450 - 350e^{\frac{\ln\frac{3}{7}}{2}t}$

$-50 = -350e^{\frac{\ln\frac{3}{7}}{2}t}$

$\frac{1}{7} = e^{\frac{\ln\frac{3}{7}}{2}t}$

$\ln\frac{1}{7} = \ln e^{\frac{\ln\frac{3}{7}}{2}t}$

$\ln\frac{1}{7} = \frac{\ln\frac{3}{7}}{2}t$

$2\ln\frac{1}{7} = \ln\frac{3}{7}\,t$

$t = \frac{2\ln\frac{1}{7}}{\ln\frac{3}{7}}$ (Use calculator to evaluate)

$t = 4.593$

The population reaches 400,000 in the middle of 1994.

10. $\frac{dR}{dt} = kR^2$

$\frac{dR}{R^2} = k\,dt$

$\int \frac{dR}{R^2} = \int k\,dt$

$-\frac{1}{R} + C_1 = kt + C_2$

$-\frac{1}{R} = kt + C$

$\left(\int \frac{dR}{R^2} \quad \int R^{-2}\,dR \quad \frac{R^{-1}}{-1} + C \quad -\frac{1}{R} + C \right)$

Find the value of C: $t=0$, $R=100$

$-\frac{1}{100} = k(0) + C$

$C = -\frac{1}{100}$ $\quad \therefore \quad -\frac{1}{R} = kt - \frac{1}{100}$

Find the value of k: $t=1$, $R=80$

$-\frac{1}{80} = k(1) - \frac{1}{100}$

$-\frac{1}{80} = k - \frac{1}{100}$

$k = -\frac{1}{80} + \frac{1}{100}$

$k = \frac{-5}{400} + \frac{4}{400}$

$k = -\frac{1}{400}$ $\quad \therefore \quad -\frac{1}{R} = -\frac{1}{400}t - \frac{1}{100}$

A. Find R when $t=6$

$-\frac{1}{R} = -\frac{1}{400}(6) - \frac{1}{100}$

$-\frac{1}{R} = -\frac{6}{400} - \frac{4}{400}$

$-\frac{1}{R} = \frac{-10}{400}$

$-10R = -400$

$R = 40$

After 6 days, there are 40 grams remaining.

B. Find t when $R = 10$.

$$-\frac{1}{R} = -\frac{1}{400}t - \frac{1}{100}$$

$$400 \cdot -\frac{1}{10} = 400 \cdot -\frac{1}{400}t - \frac{1}{100} \cdot 400$$

$$-40 = -t - 4$$

$$-36 = -t$$

$$t = 36$$

Ten grams will be left after 36 days.

11. $\frac{dP}{dt} = K\sqrt{P}$

$\frac{dP}{\sqrt{P}} = K\,dt$

$\int \frac{dP}{\sqrt{P}} = \int K\,dt$

$2\sqrt{P} + C_1 = Kt + C_2$

$2\sqrt{P} = Kt + C$

$\int \frac{dP}{\sqrt{P}}$

$\int P^{-\frac{1}{2}}\,dP$

$2P^{\frac{1}{2}} + C$

Find the value of C: $t=0$, $P=2500$

$2\sqrt{2500} = K(0) + C$

$2(50) = C$

$C = 100 \quad \therefore \quad 2\sqrt{P} = Kt + 100$

Find the value of K: $t=5$, $P=3600$

$2\sqrt{3600} = K(5) + 100$

$2(60) = 5K + 100$

$120 = 5K + 100$

$20 = 5K$

$K = 4 \quad \therefore \quad 2\sqrt{P} = 4t + 100$

Find P when $t=10$

$2\sqrt{P} = 4(10) + 100$

$2\sqrt{P} = 140$

$\sqrt{P} = 70$

$P = 4900$

The population after 10 years is 4900.