AP Calculus

REVIEW FOR 3rd QUARTER MIDTERM

Name___________________________________

Seat # ________ Date ____________________

Differential equations

In exercises 1-2, prove that y is a solution of the differential equation.

1. $y'+3y=0$; with $y=Ce^{-3x}$

2. $x^3y'''+x^2y''-3xy'-3y=0$; with $y=Cx^3$

In exercises 3-6, find the general solution to the differential equation. Can you write your solution in explicit form? (that is, can you solve your solution for y?)

3. $x\dfrac{dy}{dx}=y$

4. $y'\sec x=2y$

5. $y(1+x^3)\dfrac{dy}{dx}+x^2(1+y^2)=0$

6. $\cos x\,dy-y\,dx=0$

In exercises 7-9, find the particular solution $y=f(x)$ of the differential equation that satisfies the given condition.

7. $\dfrac{dy}{dx}=y^2+1$, with $y(1)=0$

8. $x\dfrac{dy}{dx}=\sqrt{1-y^2}$, with $y(1)=\frac{1}{2}$

9. $xdy-(2x+1)e^{-y}dx=0$ with $y(1)=2$

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10. The number of bacteria in a culture increases from 5,000 to 15,000 in 10 hours. Assume that the rate of increase is proportional to the number of bacteria present.
   a) Write a differential equation relating the number of bacteria, P, and the time in hours, t. Solve the differential equation to find a formula for the number of bacteria present in the culture at any time t.
   b) Use your formula to estimate the number of bacteria at the end of 20 hours.
   c) When will the number be 50,000?

11. If the temperature is constant, then the rate of change of barometric pressure p with respect to altitude h is proportional to p.
   a) Write a differential equation relating the barometric pressure, p, and the altitude, h.
   b) If p = 30 in. at sea level and p = 29 in. when h = 1000 ft, solve your differential equation to find a formula for the barometric pressure as a function of the altitude.
   c) Use your formula to find the pressure at an altitude of 5,000 ft.

SEE OTHER SIDE

12. The temperature of a hot liquid (such as a cup of cocoa) decreases at a rate proportional to the difference between the temperature of the liquid itself and the temperature of the room where it is cooling off. Therefore, if the temperature of the room is 70°F, T is the temperature of the cocoa, and t is the time, we have $\frac{dT}{dt} = k \cdot (T - 70)$.

a) Assume that the initial temperature of the cocoa was 200°F and after 5 minutes it went down to 170°F. Solve the given differential equation to find a formula for the temperature of the cocoa, T, as a function of the time it cooled down, t.

b) It is safe to drink cocoa when its temperature is 120°F or less. How long will it take for the cocoa to cool down to a safe temperature?

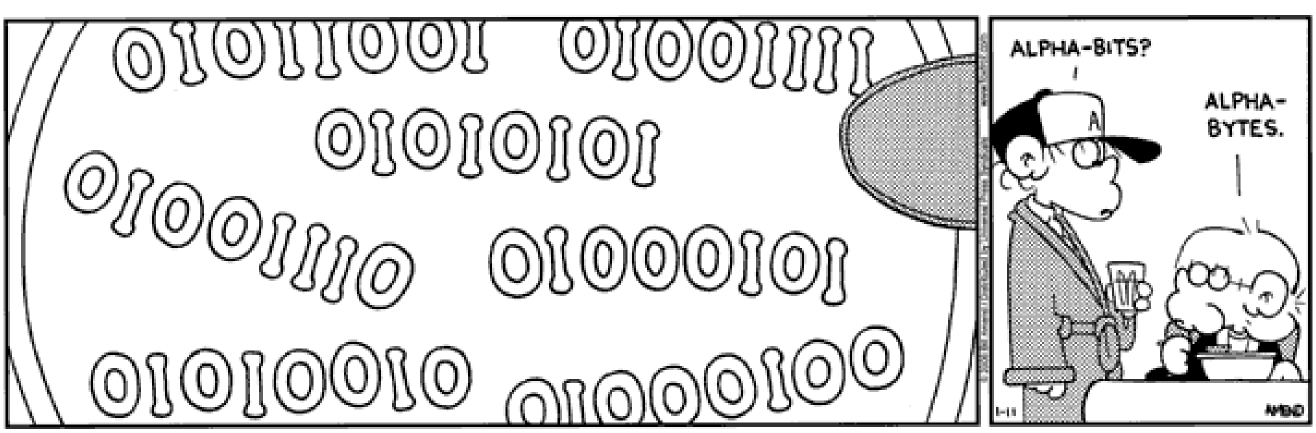
13. The wolf population P in a certain state has been growing at a rate proportional to the **cube root** of the population size. The population was estimated at 1000 in 1980 and at 1700 in 1990. Let t represent the number of years since 1980.

a) Write the differential equation relating both P and t.

b) Solve the differential equation.

c) When will the wolf population reach 4000?

Challenge! The volume V of a melting snowball decreases at a rate proportional to the surface area S of the ball. If the radius of the ball at $t = 0$ is $r = 2$ and at $t = 10$ is $r = 0.5$, find an expression for the radius r as a function of the time t. Assume that the snowball is a perfect sphere. (*Hint*: write a differential equation in terms of the radius, r, and the time, t, and solve it.)



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DIFFERENTIAL EQUATIONS

ANSWER KEY

Differential equations

1. Since
$y = Ce^{-3x} \Rightarrow y' = -3Ce^{-3x}$
So:
$y' + 3y = 0 \Rightarrow \left(-3Ce^{-3x}\right) + 3\left(Ce^{-3x}\right) = 0$

2. Since
$y = Cx^3 \Rightarrow y' = 3Cx^2 \Rightarrow y'' = 6Cx \Rightarrow y''' = 6C$
So:
$x^3 y''' + x^2 y'' - 3xy' - 3y = 0$
$x^3(6C) + x^2(6Cx) - 3x \cdot \left(3Cx^2\right) - 3 \cdot \left(Cx^3\right) = 0$

3. $y = Ax$ or $\ln|y| = \ln|x| + C$

4. $y = Ae^{2\sin x}$ or $\ln|y| = 2\sin x + C$

5. $\frac{1}{2}\ln\left(1 + y^2\right) = -\frac{1}{3}\ln\left|1 + x^3\right| + C$ or $y = \sqrt{A\left(1 + x^3\right)^{-2/3} - 1}$ or $y = -\sqrt{A\left(1 + x^3\right)^{-2/3} - 1}$

6. $y = A \cdot (\sec x + \tan x)$ or $\ln|y| = \ln|\sec x + \tan x| + C$

7. $y = \tan(x - 1)$

8. $y = \sin\left(\ln x + \frac{\pi}{6}\right)$

9. $e^y = 2x + \ln|x| + e^2 - 2$
or $y = \ln\left(2x + \ln|x| + e^2 - 2\right)$

10.
a) $\frac{dP}{dt} = k \cdot P \Rightarrow P = Ae^{kt} \Rightarrow P = 5000e^{kt}$; where $k = \frac{\ln 3}{10} \approx 0.110$
b) $t = 20$ hours $\Rightarrow 45{,}000$ bacteria
c) $50{,}000$ bacteria $\Rightarrow 50{,}000 = 5000e^{kt} \Rightarrow t \approx 20.959$ hours

11.
a) $\frac{dp}{dh} = k \cdot p$
b) $p = Ae^{kh} \Rightarrow p = 30e^{kh}$; where $k = \frac{1}{1000}\ln\left(\frac{29}{30}\right) \approx -3.39 \times 10^{-5}$
c) $h = 5{,}000$ ft $\Rightarrow 25.322$ in

12.

a) $\frac{dT}{dt} = k \cdot (T - 70) \Rightarrow \frac{dT}{T - 70} = k \cdot dt \Rightarrow \ln|T - 70| = kt + C \Rightarrow T = 70 + Ae^{kt}$

Using the conditions: $200 = 70 + Ae^{0} \Rightarrow A = 130$ and $170 = 70 + 130e^{5k} \Rightarrow k = \frac{1}{5}\ln\left(\frac{10}{13}\right)$

b) Set up equation: $120 = 70 + 130e^{kt} \Rightarrow t \approx 18.210$ minutes

13.

a) $\frac{dP}{dt} = k \cdot P^{\frac{1}{3}}$

b) $dP = k \cdot P^{\frac{1}{3}} \cdot dt \Rightarrow P^{-1/3} \cdot dP = k \cdot dt \Rightarrow \frac{3}{2}P^{\frac{2}{3}} = kt + C$

$\frac{3}{2}P^{\frac{2}{3}} = 6.366t + 150 \Rightarrow P = (4.244t + 100)^{\frac{3}{2}}$

c) $P = 4000 \Rightarrow 4000 = (4.244t + 100)^{\frac{3}{2}} \Rightarrow t = 35 \Rightarrow$ year 2015

Challenge!

$$\left.\begin{array}{l} \frac{dV}{dt} = k \cdot 4\pi r^2 \\ V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} \end{array}\right\} \Rightarrow \frac{dV}{dt} = k \cdot 4\pi r^2 = 4\pi r^2 \cdot \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = k \Rightarrow r = -\frac{3}{20}t + 2$$