AP Calculus AB

Practice Test 1 (Differentiation)

Part I: Multiple Choice: No Calculator

1) If $f(x) = \frac{x^2-9}{x+3}$ has a removable discontinuity at $x = -3$, then $f(-3) =$

A) 3
B) -3
C) 0
D) 6
[✓ E) -6]

This function has a discontinuity where the denominator is zero. i.e., at $x = -3$. If this discontinuity is removed, then, we have:

$$f(x) = \frac{x^2-9}{x+3} = x - 3$$

So, $f(-3) = -3 - 3 = -6$

Answer: E

2) $\lim_{x\to\infty} \frac{10^8x^5 + 10^6x^4 + 10^4x^2}{10^9x^6 + 10^7x^5 + 10^5x^3}$

[✓ A) 0]
B) 1
C) -1
D) $\frac{1}{10}$
D) $-\frac{1}{10}$

For this rational function, the highest power of x takes over when calculating a limit. Since the x^6 term exists in the denominator and not in the numerator, the ratio of the two polynomials gets smaller and smaller as $x \to 0$, eventually approaching 0.

Alternatively, apply L'Hospital's rule repeatedly (5 times) until you have a constant in the numerator and a linear term in the denominator. The ratio can then be seen to approach zero as $x \to \infty$.

Answer: A

3) If $f(x) = \sqrt{4\sin 2x + 2}$, then $f'(0) =$

A) -2
B) 0
C) $\sqrt{2}$
[✓ D) $2\sqrt{2}$]
E) 1

$$f(x) = (4\sin 2x + 2)^{1/2}$$

$$f'(x) = \frac{1}{2}(4\sin 2x + 2)^{-1/2} \cdot (4\cos 2x) \cdot 2$$

$$= \frac{(4\cos 2x)}{\sqrt{4\sin 2x + 2}}$$

$$f'(0) = \frac{(4\cos 0)}{\sqrt{4\sin 0 + 2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

Answer: D

4) Let f and g be differentiable functions such that

$$f(1) = 4,\ g(1) = 3,\ f'(3) = -5$$

$$f'(1) = -4,\ g'(1) = -3,\ g'(3) = 2$$

If $h(x) = f(g(x))$, then $h'(1) =$

A) -9
[✓ B) 15]
C) 0
D) -5
E) -12

Use the chain rule in Lagrange (Prime) Notation.

$$h'(x) = f'\big(g(x)\big) \cdot g'(x), \qquad \text{where: } h = f \circ g$$

$$h'(1) = f'\big(g(1)\big) \cdot g'(1) = f'(3) \cdot g'(1) = (-5)\cdot(-3) = 15$$

Answer: B

5) What is $\lim_{h\to 0} \frac{\cos\left(\frac{\pi}{2}+h\right)-\cos\left(\frac{\pi}{2}\right)}{h}$?

[✓ A) -1]
B) $-\frac{\sqrt{2}}{2}$
C) 0
D) 1
E) The limit does not exist.

This limit is the definition of a derivative.

$$\lim_{h\to 0} \frac{\cos\left(\frac{\pi}{2}+h\right)-\cos\left(\frac{\pi}{2}\right)}{h} = \frac{d}{dx}(\cos x)\Big|_{x=\frac{\pi}{2}}$$

$$= (-\sin x)\Big|_{x=\frac{\pi}{2}} = -\sin\frac{\pi}{2} = -1$$

Answer: A

6) If $f(x) = \sin^2(3 - x)$, then $f'(0) =$

(A) $-2\cos 3$
[✓ (B) $-2\sin 3\cos 3$]
(C) $6\cos 3$
(D) $2\sin 3\cos 3$
(E) $6\sin 3\cos 3$

$$f(x) = \sin^2(3-x)$$

$$f'(x) = 2\cdot\sin(3-x) \cdot \frac{d}{dx}[\sin(3-x)]$$

$$= 2 \cdot \sin(3-x) \cdot \cos(3-x) \cdot \frac{d}{dx}(3-x)$$

$$= 2 \cdot \sin(3-x) \cdot \cos(3-x) \cdot (-1)$$

$$f'(0) = -2 \cdot \sin(3) \cdot \cos(3)$$

Answer: B

7) If $\lim_{x\to 2}\frac{f(x)}{x-2} = f'(2) = 0$, which of the following must be true?

I. $f(2) = 0$
II. $f(x)$ is continuous at $x = 2$
III. $f(x)$ has a horizontal tangent line at $x = 2$

A) I only
B) II only
C) I and II only
D) II and III only
[✓ E) I, II, and III]

Check the conditions one at a time.

I. $$f'(x) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

$$f'(2) = \lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$$

$$\text{if,}\quad f'(2) = \lim_{x\to 2}\frac{f(x)}{x-2} = \lim_{x\to 2}\frac{f(x)-0}{x-2}$$

it must be true that $f(2) = 0$ **I is TRUE**

II. f **is differentiable at** $x = 2$**, and so must be continuous at** $x = 2$**.**
II is TRUE

III. The first derivative provides the slope of the tangent line. Since $f'(2) = 0$**, there must be a tangent of slope 0 (horizontal) at** $x = 2$**.**

III is TRUE

Answer: E

8) If $f(x) = x - 1$ and $g(x) = x^2 + 1$, then $f(g(x)) = g(f(x))$ when $x =$

A) $-\frac{1}{2}$
B) $\frac{1}{2}$
C) -1
[✓ D) 1]
E) 0

$$f(g(x)) = f(x^2+1) = x^2+1-1 = x^2$$

$$g(f(x)) = g(x-1) = (x-1)^2+1 = x^2-2x+1+1 = x^2-2x+2$$

Then, set: $f(g(x)) = g(f(x))$

$$x^2 = x^2 - 2x + 2$$

$$2x = 2$$

$$x = 1$$

Answer: D

9) Which of the following are equal to cos (2x)?

I. $\cos^2 x - \sin^2 x$
II. $\cos^2 x + \sin^2 x$
III. $2\cos^2 x - 1$

A) I only
B) II only
[✓ C)] I and III
D) all
E) none

Here are the trigonometry double-angle formulas:

$$\sin 2A = 2 \sin A \cos A$$

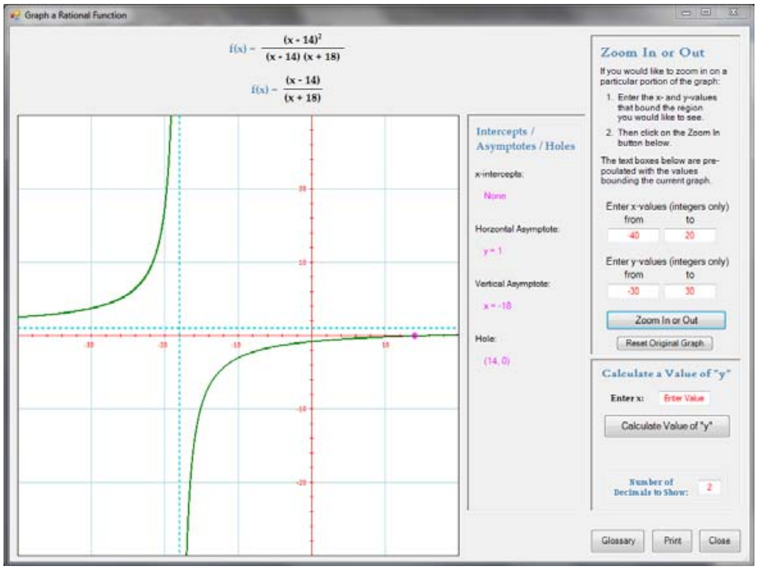
$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= 1 - 2\sin^2 A \\ &= 2\cos^2 A - 1\end{aligned}$$

$$\tan 2A = \frac{2 \tan A}{1 - tan^2 A}$$

Answer: C

10) What are the asymptotes of $f(x) = \frac{(x-14)^2}{(x+18)(x-14)}$?

A) horizontal at $y = 0$, no vertical
B) horizontal at $y = 0$, vertical at $x = -18$
C) horizontal at $y = 0$, vertical at $x = -18$ and $x = 14$
[✓ D)] horizontal at $y = 1$, vertical at $x = -18$
E) horizontal at $y = 1$, vertical at $x = -18$ and $x = 14$



x-intercepts: None

Horizontal Asymptote: y = 1

Vertical Asymptote: x = -18

Hole: (14, 0)

If you would like to zoom in on a particular portion of the graph:

1. Enter the x- and y-values that bound the region you would like to see.
2. Then click on the Zoom In button below.

The text boxes below are pre-poulated with the values bounding the current graph.

Enter x-values (integers only) from -40 to 20

Enter y-values (integers only) from -30 to 30

Enter x: Enter Value

Number of Decimals to Show: 2

Image from the Algebra App available at www.mathguy.us

This is a rational function with a hole at $x = 14$ and a vertical asymptote at $x = -18$.

- A **hole** exists if the multiplicity of a root in the denominator $\le$ the multiplicity of the same root in the numerator.
- A **vertical asymptote** exists if the multiplicity of a root in the denominator $>$ the multiplicity of the same root in the numerator.

To get the horizontal asymptote, calculate:

$$\lim_{x\to\infty} \frac{(x-14)^2}{(x+18)(x-14)} =$$

$$\lim_{x\to\infty} \frac{x^2 - 28x + 196}{x^2 + 4x - 252}$$

Using L'Hospital's Rule, take the derivatives of the numerator and denominator twice, to get $f(x) = 1$.

Answer: D

Part II: Multiple Choice: Calculators Are Allowed

11) The equation of the tangent line to the curve $x^2 + y^2 = 169$ at the point (5, -12) is

A) $5y - 12x = -120$
B) $5x - 12y = 119$
[✓ C)] $5x - 12y = 169$
D) $12x + 5y = 0$
E) $12x + 5y = 169$

Use implicit differentiation:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(169)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{5}{-12} = \frac{5}{12}$$

So, the slope of the tangent line, $m = \frac{5}{12}$

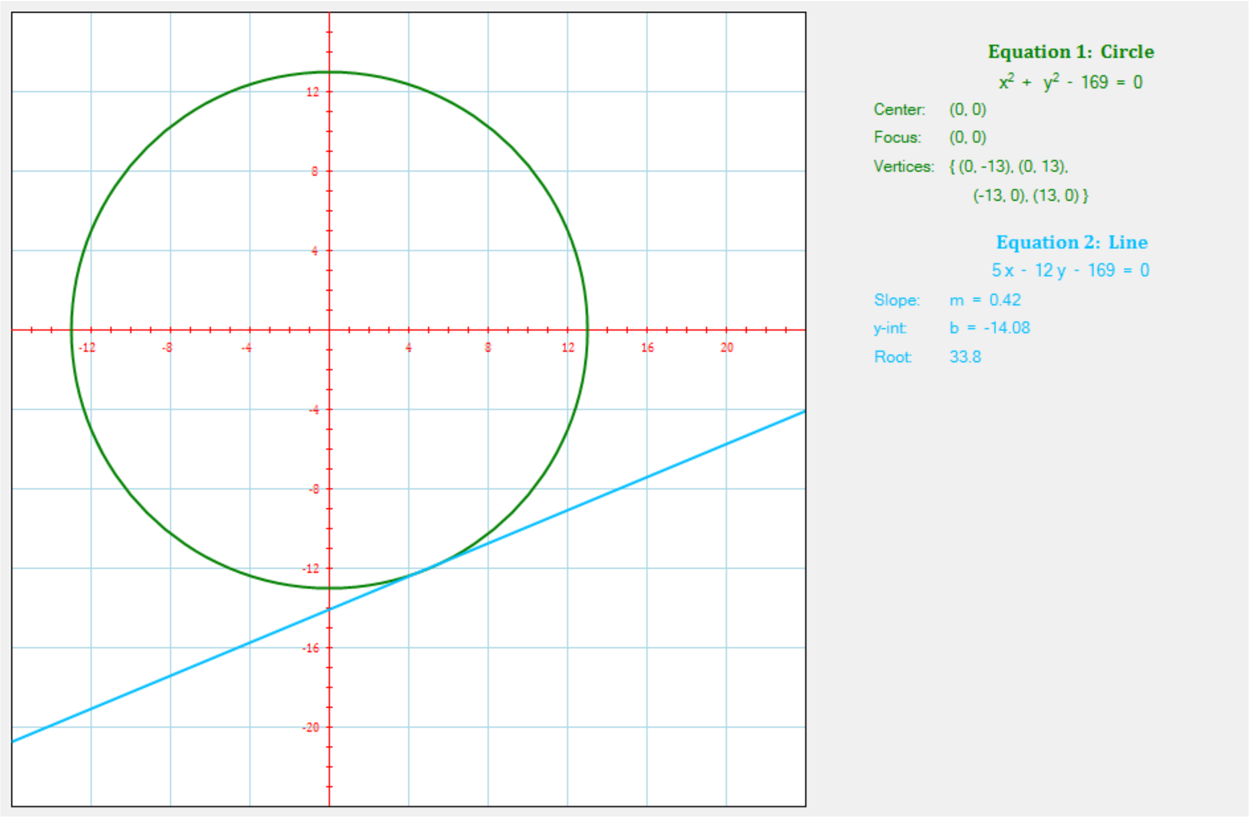
Then, using the point-slope form of a line:

$$y + 12 = \frac{5}{12}(x - 5)$$

$$12y + 144 = 5x - 25$$

$$169 = 5x - 12y$$

Answer: C



Equation 1: Circle
$x^2 + y^2 - 169 = 0$
Center: (0, 0)
Focus: (0, 0)
Vertices: { (0, -13), (0, 13), (-13, 0), (13, 0) }

Equation 2: Line
$5x - 12y - 169 = 0$
Slope: m = 0.42
y-int: b = -14.08
Root: 33.8

Image from the Algebra App available at www.mathguy.us

12) A point moves along the curve $y = x^2 + 1$ in such a way that when $x = 4$, the x-coordinate is increasing at the rate of 5 ft/sec. At what rate is the y-coordinate changing at that time?

A) 80 ft/sec
B) 45 ft/sec
C) 32 ft/sec
D) 85 ft/sec
[✓ E)] 40 ft/sec

This is a related rates problem with: $\frac{dx}{dt} = 5$ **when** $x = 4$.

$$\frac{d}{dt}(y) = \frac{d}{dt}(x^2 + 1)$$

$$\frac{dy}{dt} = 2x \cdot \frac{dx}{dt}$$

Substitute values for $x = 4$ **and** $\frac{dx}{dt} = 5$ **to get:**

$$\frac{dy}{dt} = 2 \cdot (4) \cdot (5) = 40 \text{ ft/sec}$$

Answer: E

13) The equation of the line tangent to the curve $y = \frac{kx + 8}{k + x}$ at $x = -2$ is $y = x + 4$. What is the value of k?

A) -3
B) -1
C) 1
[✓ D)] 3
E) 4

Use either the product or quotient rule to calculate $\frac{dy}{dx}$**. Note also that** $\frac{dy}{dx} = 1$ **because the slope of the line** $y = x + 4$ **is 1.**

$$\frac{dy}{dx} = \frac{(k + x) \cdot \frac{d}{dx}(kx + 8) - (kx + 8) \cdot \frac{d}{dx}(k + x)}{(k + x)^2}$$

$$1 = \frac{(k + x) \cdot (k) - (kx + 8) \cdot (1)}{(k + x)^2} = \frac{(k - 2) \cdot (k) - (-2k + 8) \cdot (1)}{(k - 2)^2}$$

substituting -2 for the value of x

$$(k - 2)^2 = k^2 - 2k + 2k - 8$$

$$k^2 - 4k + 4 = k^2 - 8$$

$$-4k = -12$$

$$k = 3$$

Answer: D

Part III: Free Response: No Calculators

1) Let *f* be the function given by $f(x) = \frac{x}{\sqrt{x^2-4}}$

a) Find the domain of *f*. (Write your answer in interval notation.)
b) Write an equation for each vertical asymptote to the graph of *f*.
c) Write an equation for each horizontal asymptote to the graph of *f*.
d) Find *f* '(*x*).

a. The domain of x exists where the denominator is real and non-zero.

$$x^2 - 4 > 0 \quad \Rightarrow \quad x^2 > 4 \quad \Rightarrow \quad |x| > 2$$

b. Vertical asymptotes exist where the denominator is zero.

$$x^2 - 4 = 0 \quad \Rightarrow \quad |x| = 2 \quad \Rightarrow \quad x = \pm 2$$

c. Left horizontal asymptote (easy method is to eliminate the constant):

$$y = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2-4}} = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2}} = \lim_{x\to-\infty} \frac{x}{|x|} = -1 \qquad \Rightarrow \quad y = -1$$

Right horizontal asymptote (easy method is to eliminate the constant):

$$y = \lim_{x\to+\infty} \frac{x}{\sqrt{x^2-4}} = \lim_{x\to+\infty} \frac{x}{\sqrt{x^2}} = \lim_{x\to+\infty} \frac{x}{|x|} = 1 \qquad \Rightarrow \quad y = 1$$

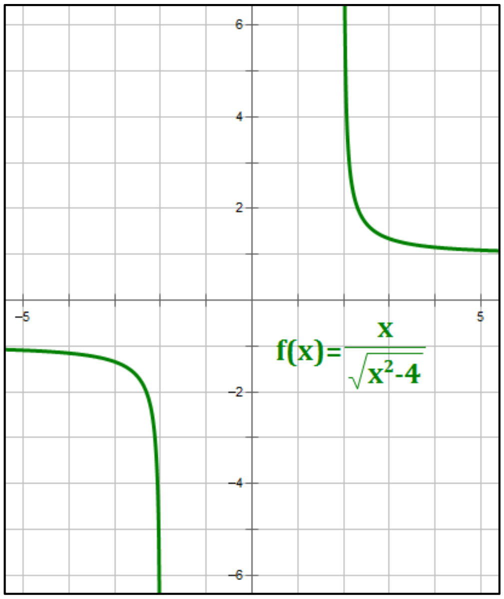
d. $$f'(x) = \frac{(x^2-4)^{1/2}\cdot\frac{d}{dx}(x) - x\cdot\frac{d}{dx}\left[(x^2-4)^{1/2}\right]}{x^2-4}$$

$$= \frac{(x^2-4)^{1/2}\cdot 1 \; - \; x\cdot\frac{1}{2}\left[(x^2-4)^{-1/2}\right]\cdot 2x}{x^2-4}$$

$$= \frac{(x^2-4)^{1/2} \; - \; x^2\cdot\left[(x^2-4)^{-1/2}\right]}{x^2-4}$$

$$= \frac{(x^2-4)^{-1/2}\cdot[(x^2-4)-x^2]}{x^2-4}$$

$$= \frac{-4}{(x^2-4)^{1/2}\cdot(x^2-4)} = \frac{-4}{(x^2-4)^{3/2}}$$



2) Given $f(x) = 8x^3$, write an equation for the line tangent to the graph of f at $x = \frac{1}{2}$.

The slope of the tangent line is equal to the derivative of the function at that point.

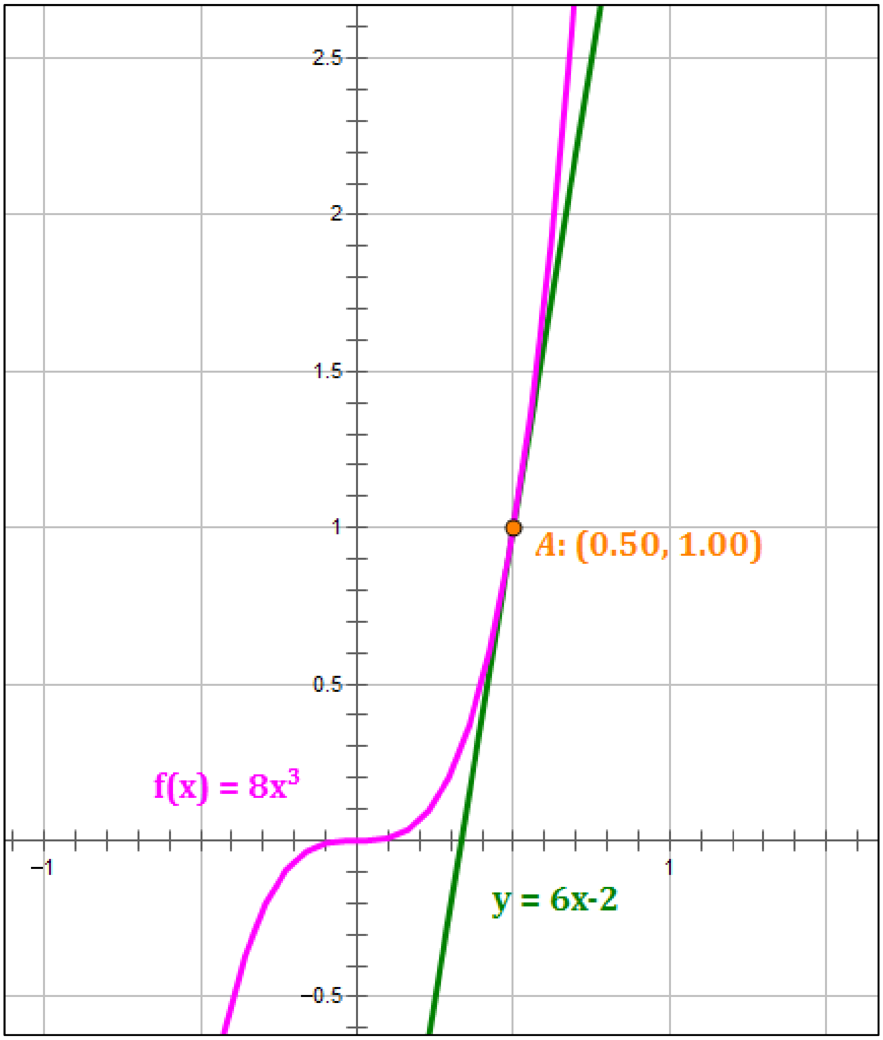
$$\frac{d}{dx}(f(x)) = 24x^2 \qquad \therefore \qquad m = 24 \cdot \left(\frac{1}{2}\right)^2 = 6$$

Next, find $f\left(\frac{1}{2}\right) = 8 \cdot \left(\frac{1}{2}\right)^3 = 1$

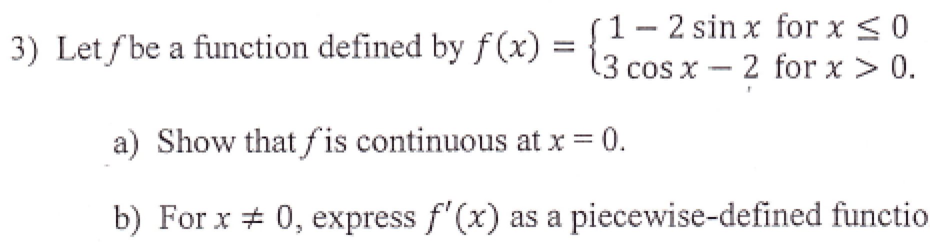
So, $\left(\frac{1}{2}, 1\right)$ **is a point on the curve.**

Then, in point-slope form, the equation is: $y - 1 = 6\left(x - \frac{1}{2}\right)$

In slope-intercept form, this is: $y = 6x - 2$



Part IV: Free Response: Calculator Portion



3) Let f be a function defined by $f(x) = \begin{cases} 1 - 2\sin x & \text{for } x \le 0 \\ 3\cos x - 2 & \text{for } x > 0. \end{cases}$

a) Show that f is continuous at $x = 0$.

b) For $x \neq 0$, express $f'(x)$ as a piecewise-defined function.

Both "piece functions" are continuous.

a. The following items are required for a piecewise function to be continuous:

- Each piece must be continuous
- The function must exist at points where the pieces "come together."
- The limits from the left and right must both equal the value of the curve at points where the pieces "come together."

Now, consider f :

Note that each of the two component functions of f is continuous. So, we need only check the point at which the two curves "come together", i.e., $x = 0$.

We require that $f(0)$ exists. It does exist and, from the top function, $f(0) = 1$.

Finally, check the limits at $x = 0$ from the left and the right:

$$\lim_{x \to 0-} (1 - 2\sin x) = 1 - 2 \cdot 0 = 1$$

$$\lim_{x \to 0+} (3\cos x - 2) = 3 \cdot 1 - 2 = 1$$

So, we see that: $\lim_{x \to 0-} \big(f(x)\big) = \lim_{x \to 0+} \big(f(x)\big) = f(0)$

Therefore, the function is continuous.

b. Take the derivative of each piece. Note that we are given $x \neq 0$.

$$\frac{d}{dx}(1 - 2\sin x) = -2\cos x \qquad \text{for } \; x < 0$$

$$\frac{d}{dx}(3\cos x - 2) = -3\sin x \qquad \text{for } \; x > 0$$

4) At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill, and this model is estimated to work for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010. $\frac{dW}{dt} = \frac{1}{25}(W - 300)$

a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

b) Find $\frac{d^2W}{dt^2}$ in terms of W.

A note on notation: W_a and $f(a)$ both refer to the value of the function at $x = a$.

a. See the image at right. We are given:

$$W_0 = f(0) = 1,400$$

$$\frac{dW}{dt} = \frac{1}{25}(W - 300)$$

Notice that the curve is close to a straight line on the interval we care about, so we can approximate the change from $t = 0$ to $t = 0.25$ as linear.

The slope of the curve at $t = 0$ is:

$$\left.\frac{dW}{dt}\right|_{W=1,400} = \frac{1}{25}(1,400 - 300)$$

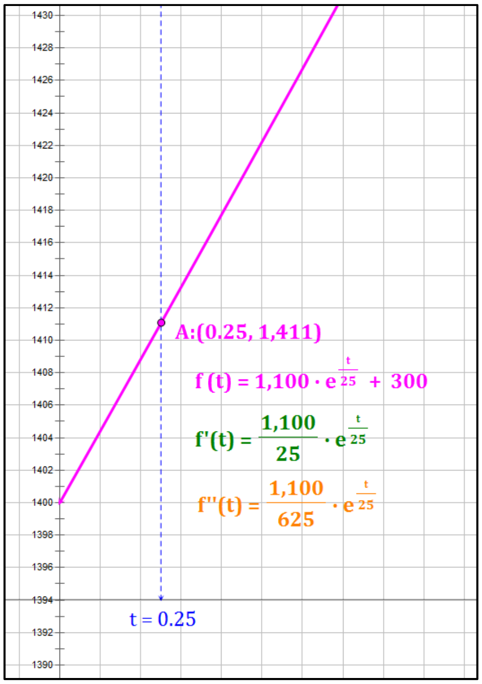
$$= 44 \text{ tons/year}$$

Estimate, then, that

$$W_1 = f(1) \sim 1,400 + 44 = 1,444 \text{ tons}$$

$$W_{0.25} = f(0.25) \sim 1,400 + (.25) \cdot (44)$$

$$\sim 1,411 \text{ tons}$$



The above curve, $f(t)$, can be derived using Integral Calculus, to which the student has not yet been exposed. Nevertheless, the illustration may be helpful in visualizing the situation.

b. $$\frac{d^2W}{dt^2} = \frac{d}{dt}\left(\frac{dW}{dt}\right) = \frac{d}{dt}\left[\frac{1}{25}(W - 300)\right]$$

$$= \frac{1}{25} \cdot \frac{dW}{dt} = \frac{1}{25} \cdot \left[\frac{1}{25}(W - 300)\right] = \frac{1}{625}(W - 300)$$