AP Calculus BC
Chapter SERIES and TAYLOR AP Exam Problems

<u>All problems are NON CALCULATOR unless otherwise indicated.</u>

1. If $S_n = \left(\frac{(5+n)^{100}}{5^{n+1}}\right)\left(\frac{5^n}{(4+n)^{100}}\right)$, to what number does the sequence $\{S_n\}$ converge?

A) $\frac{1}{5}$ B) 1 C) $\frac{3}{4}$ D) $\left(\frac{5}{4}\right)^{100}$ E) Does not converge

2. Which of the following series are convergent?

I. $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}+\cdots$

II. $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$

III. $1-\frac{1}{3}+\frac{1}{3^2}-\cdots+\frac{(-1)^{n+1}}{3^{n-1}}+\cdots$

A) I only
B) III only
C) I and III
D) II and III
E) I, II, and III

3. Which of the following series converge?

I. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n+1}$ II. $\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{3}{2}\right)^n$ III. $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$

A) I only
B) II only
C) III only
D) I and III
E) I, II, and III

4. Which of the following series diverge?

I. $\sum_{k=3}^{\infty}\frac{1}{k^2+1}$ II. $\sum_{k=1}^{\infty}\left(\frac{6}{7}\right)^k$ III. $\sum_{k=2}^{\infty}\frac{(-1)^k}{k}$

A) None
B) II only
C) III only
D) I and III
E) II and III

5. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{n}{n+2}$ II. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ III. $\sum_{n=1}^{\infty} \frac{1}{n}$

A) None
B) II only
C) III only
D) I and II
E) I and III

6. If $\lim_{b\to\infty} \int_1^b \frac{dx}{x^p}$ is finite, which of the following must be true?

A) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges
B) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges
C) $\sum_{n=1}^{\infty} \frac{1}{n^{p-2}}$ converges
D) $\sum_{n=1}^{\infty} \frac{1}{n^{p-1}}$ converges
E) $\sum_{n=1}^{\infty} \frac{1}{n^{p+1}}$ diverges

7. For what integer k, $k > 1$, will both $\sum_{n=2}^{\infty} \frac{(-1)^{kn}}{n}$ and $\sum_{n=1}^{\infty} \left(\frac{k}{4}\right)^n$ converge?

A) 6 B) 5 C) 4 D) 3 E) 2

8. For $-1 < x < 1$ if $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{2n-1}$, then $f'(x) =$

A) $\sum_{n=1}^{\infty} (-1)^{n+1} x^{2n-2}$
B) $\sum_{n=1}^{\infty} (-1)^{n} x^{2n-2}$
C) $\sum_{n=1}^{\infty} (-1)^{2n} x^{2n}$
D) $\sum_{n=1}^{\infty} (-1)^{n} x^{2n}$
E) $\sum_{n=1}^{\infty} (-1)^{n+1} x^{2n}$

9. The coefficient for x^3 in the Taylor series for e^{3x} about $x = 0$ is

A) $\frac{1}{6}$ B) $\frac{1}{3}$ C) $\frac{1}{2}$ D) $\frac{3}{2}$ E) $\frac{9}{2}$

10. Which of the following is a series expansion of $\sin(2x)$?

A) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} + \cdots$

B) $2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots + \frac{(-1)^{n-1} (2x)^{2n-1}}{(2n-1)!} + \cdots$

C) $-\frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots + \frac{(-1)^n (2x)^{2n}}{(2n)!} + \cdots$

D) $\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$

E) $2x + \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \cdots + \frac{(2x)^{2n-1}}{(2n-1)!} + \cdots$

11. The coefficient of x^6 in the Taylor series expansion about $x = 0$ for $f(x) = \sin\left(x^2\right)$ is

A) $-\frac{1}{6}$ B) 0 C) $\frac{1}{120}$ D) $\frac{1}{6}$ E) 1

12. What is the approximation of the value of $\sin 1$ obtained by using the fifth-degree Taylor polynomial about $x = 0$ for $\sin x$?

A) $1 - \frac{1}{2} + \frac{1}{24}$

B) $1 - \frac{1}{2} + \frac{1}{4}$

C) $1 - \frac{1}{3} + \frac{1}{5}$

D) $1 - \frac{1}{4} + \frac{1}{8}$

E) $1 - \frac{1}{6} + \frac{1}{120}$

13. If $\sum_{n=0}^{\infty} a_n x^n$ is a Taylor series that converges to $f(x)$ for all real x, then $f'(1) =$

A) 0

B) a_1

C) $\sum_{n=0}^{\infty} a_n$

D) $\sum_{n=1}^{\infty} n a_n$

E) $\sum_{n=1}^{\infty} n a_n x^{n-1}$

14. (**CALCULATOR PROBLEM**) The graph of the function represented by the Maclaurin series $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^n}{n!} + \cdots$ intersects the graph of $y = x^3$ at $x =$

A) 0.773 B) 0.865 C) 0.929 D) 1.000 E) 1.857

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15. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$ converges?

A) $-1 \le x < 1$ B) $-1 \le x \le 1$ C) $0 < x < 2$ D) $0 \le x < 2$ E) $0 \le x \le 2$

16. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges?

A) $-1 \le x \le 1$ B) $-1 < x \le 1$ C) $-1 \le x < 1$ D) $-1 < x < 1$ E) all real x

17. The interval of convergence of $\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n}$ is

A) $-3 < x \le 3$ B) $-3 \le x \le 3$ C) $-2 < x < 4$ D) $-2 \le x < 4$ E) $0 \le x \le 2$

18. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n}}$ converges?

A) $-3 < x < -1$ B) $-3 \le x < -1$ C) $-3 \le x \le -1$ D) $-1 \le x < 1$ E) $-1 \le x \le 1$

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19. (1990 BC5) Let f be the function defined by $f(x) = \frac{1}{x-1}$.

(a) Write the first four terms and the general term of the Taylor series expansion of $f(x)$ about $x = 2$.

(b) Use the result from part (a) to find the first four terms and the general term of the series expansion about $x = 2$ for $\ln|x-1|$.

(c) Use the series in part (b) to compute a number that differs from $\ln\frac{3}{2}$ by less than 0.05. Justify your answer.

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20. (1992 BC6) Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^p \ln n}$, where $p>0$.

   (a) Show that the series converges for $p > 1$.
   (b) Determine whether the series converges or diverges for $p = 1$. Show your analysis.
   (c) Show that the series diverges for $0 \le p < 1$.

21. (1995 BC4) Let f be a function that has derivatives of all orders for all real numbers. Assume $f(1)=3,\ f'(1)=-2,\ f''(1)=2,\ \textit{and}\ \ f'''(1)=4$.

   (a) Write the second-degree Taylor polynomial for f about $x = 1$ and use it to approximate $f(0.7)$.
   (b) Write the third-degree Taylor polynomial for f about $x = 1$ and use it to approximate $f(1.2)$.
   (c) Write the second-degree Taylor polynomial for f', the derivative of f, about $x = 1$ and use it to approximate $f'(1.2)$.

22. (1997 BC2) Let $P(x)=7-3(x-4)+5(x-4)^2-2(x-4)^3+6(x-4)^4$ be the fourth-degree Taylor polynomial for the function f about $x = 4$. Assume f has derivatives of all orders for all real numbers.

   (a) Find $f(4)$ and $f'''(4)$.
   (b) Write the second-degree Taylor polynomial for f' about $x = 4$ and use it to approximate $f'(4.3)$.
   (c) Write the fourth-degree Taylor polynomial for $g(x)=\int_4^x f(t)dt$ about 4.
   (d) Can $f(3)$ be determined from the information given? Justify your answer.

23. (1998 BC3) Let f be a function that has derivatives of all orders for all real numbers. Assume $f(0)=5,\ f'(0)=-3,\ f''(0)=1,\ \textit{and}\ \ f'''(0)=4$.

   (a) Write the third-degree Taylor polynomial for f about $x = 0$ and use it to approximate $f(0.2)$.
   (b) Write the fourth-degree Taylor polynomial for g, where $g(x)=f\left(x^2\right)$, about $x = 0$.
   (c) Write the third-degree Taylor polynomial for h, where $h(x)=\int_0^x f(t)dt$, about $x = 0$.
   (d) Let h be defined as in part (c). Given that $f(1)=3$, either find the exact value of $h(1)$ or explain why it cannot be determined.

24. (1999 BC4) The function *f* has derivatives of all orders for all real numbers *x*. Assume that $f(2)=-3,\ f'(2)=5,\ f''(2)=3,\ and\ f'''(2)=-8$.

(a) Write the third-degree Taylor polynomial for *f* about *x* = 2 and use it to approximate $f(1.5)$.

(b) The fourth derivative of *f* satisfies the inequality $\left|f^4(x)\right| \le 3$ for all *x* in the closed interval [1.5, 2]. Use the Lagrange error bound on the approximation to $f(1.5)$ found in part (a) to explain why $f(1.5) \ne -5$.

(c) Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x)=f\left(x^2+2\right)$ about *x* = 0. Use *P* to explain why *g* must have a relative minimum at *x* = 0.

25. (2001 BC6) A function *f* is defined by $f(x)=\frac{1}{3}+\frac{2}{3^2}x+\frac{3}{3^3}x^2+\cdots+\frac{n+1}{3^{n+1}}x^n+\cdots$ for all *x* in the interval of convergence of the given power series.

(a) Find the interval of convergence for this power series. Show the work that leads to your answer.

(b) Find $\lim\limits_{x\to 0}\dfrac{f(x)-\frac{1}{3}}{x}$.

(c) Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^1 f(x)dx$.

(d) Find the sum of the series determined in part (c).

26. (2002 BC6) The Maclaurin series for the function *f* is given by

$$f(x)=\sum_{n=0}^{\infty}\frac{(2x)^{n+1}}{n+1}=2x+\frac{4x^2}{2}+\frac{8x^3}{3}+\frac{16x^4}{4}+\cdots+\frac{(2x)^{n+1}}{n+1}+\cdots$$

(a) Find the interval of convergence of the Maclaurin series for *f*. Justify your answer.

(b) Find the first four terms and the general term of the Maclaurin series for $f'(x)$.

(c) Use the Maclaurin series you found in part (b) to find the value of $f'\left(-\frac{1}{3}\right)$.

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27. (2002B BC6) The Maclaurin series for $\ln\left(\frac{1}{1-x}\right)$ is $\sum_{n=1}^{\infty} \frac{x^n}{n}$ with interval of convergence $-1 \le x < 1$.

(a) Find the Maclaurin series for $\ln\left(\frac{1}{1+3x}\right)$ and determine the interval of convergence.

(b) Find the value of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

(c) Give a value of p such that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ diverges. Give reasons why your value of p is correct.

(d) Give a value of p such that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges. Give reasons why your value of p is correct.

28. (2003 BC6) The function f is defined by the power series

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots+\frac{(-1)^n x^{2n}}{(2n+1)!}+\cdots \text{ for all real numbers } x.$$

(a) Find $f'(0)$ and $f''(0)$. Determine whether f has a local maximum, a local minimum, or neither at $x = 0$. Give a reason for your answer.

(b) Show that $1-\frac{1}{3!}$ approximates $f(1)$ with error less than $\frac{1}{100}$.

(c) Show that $y = f(x)$ is a solution to the differential equation $xy' + y = \cos x$.

29. (2003B BC6) The function f has a Taylor series about $x = 2$ that converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 2$ is given by $f^{(n)}(2)=\frac{(n+1)!}{3^n}$ for $n \ge 1$, and $f(2) = 1$.

(a) Write the first four terms and the general term of the Taylor series for f about $x = 2$.

(b) Find the radius of convergence for the Taylor series for f about $x = 2$. Show the work that leads to your answer.

(c) Let g be a function satisfying $g(2) = 3$ and $g'(x) = f(x)$ for all x. Write the first four terms and the general tem of the Taylor series for g about $x = 2$.

(d) Does the Taylor series for g as defined in part (c) converge at $x = -2$? Give a reason for your answer.

30. (2004 BC6) Let *f* be the function given by $f(x)=\sin\left(5x+\frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for *f* about $x = 0$.

(a) Find $P(x)$.
(b) Find the coefficient of x^{22} in the Taylor series for *f* about $x = 0$.
(c) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right)-P\left(\frac{1}{10}\right)\right|<\frac{1}{100}$.
(d) Let *G* be the function given by $G(x)=\int_0^x f(t)dt$. Write the third-degree Taylor polynomial for *G* about $x = 0$.

31. (2004B BC2) Let *f* be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for *f* about $x = 2$ is given by $T(x)=7-9(x-2)^2-3(x-2)^3$.

(a) Find $f(2)$ and $f''(2)$.
(b) Is there enough information given to determine whether *f* has a critical point at $x = 2$? If not, explain why not. If so, determine whether $f(2)$ is a relative maximum, a relative minimum, or neither, and justify your answer.
(c) Use $T(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether *f* has a critical point at $x = 0$? If not, explain why not. If so, determine whether $f(0)$ is a relative maximum, a relative minimum, or neither, and justify your answer.
(d) The fourth derivative of *f* satisfies the inequality $\left|f^{(4)}(x)\right|\le 6$ for all *x* in the closed interval [0, 2]. Use the Lagrange error bound on the approximation to $f(0)$ found in part (c) to explain why $f(0)$ is negative.

32. (2005 BC6) Let *f* by a function with derivatives of all orders and for which $f(2) = 7$. When *n* is odd, the *n*th derivative of *f* at $x = 2$ is 0. When *n* is even and $n\ge 2$, the *n*th derivative of *f* at $x = 2$ is given by $f^{(n)}(2)=\frac{(n-1)!}{3^n}$.

(a) Write the sixth-degree Taylor polynomial for *f* about $x = 2$.
(b) In the Taylor series for *f* about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n\ge 1$?
(c) Find the interval of convergence of the Taylor series for *f* about $x = 2$. Show the work that leads to your answer.

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33. (2005B BC3) The Taylor series about $x = 0$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 0$ is given by $f^{(n)}(0)=\dfrac{(-1)^{n+1}(n+1)!}{5^n(n-1)^2}$ for $n\geq 2$. The graph of f has a horizontal tangent line at $x = 0$, and $f(0) = 6$.

(a) Determine whether f has a relative maximum, a relative minimum, or neither at $x = 0$. Justify your answer.

(b) Write the third-degree Taylor polynomial for f about $x = 0$.

(c) Find the radius of convergence of the Taylor series for f about $x = 0$. Show the work that leads to your answer.

34. (2006 BC6) The function f is defined by the power series:

$f(x)=-\dfrac{x}{2}+\dfrac{2x^2}{3}-\dfrac{3x^3}{4}+\cdots+\dfrac{(-1)^n nx^n}{n+1}+\cdots$ for all real numbers x for which the series converges. The function g is defined by the power series:

$g(x)=1-\dfrac{x}{2!}+\dfrac{x^2}{4!}-\dfrac{x^3}{6!}+\cdots+\dfrac{(-1)^n x^n}{(2n)!}+\cdots$ for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The graph of $y=f(x)-g(x)$ passes through the point $(0,-1)$. Find $y'(0)$ and $y''(0)$. Determine whether y has a relative minimum, a relative maximum, or neither at $x = 0$. Give a reason for your answer.

35. (2006B BC6) The function f is defined by $f(x)=\dfrac{1}{1+x^3}$. The Maclaurin series for f is given by $1-x^3+x^6-x^9+\cdots+(-1)^n x^{3n}+\cdots$, which converges to $f(x)$ for $-1<x<1$.

(a) Find the first three nonzero terms and the general term for the Maclaurin series for $f'(x)$.

(b) Use your results fron part (a) to find the sum of the infinite series $-\dfrac{3}{2^2}+\dfrac{6}{2^5}-\dfrac{9}{2^8}+\cdots+(-1)^n\dfrac{3n}{2^{3n-1}}+\cdots$.

(c) Find the first four nonzero terms and the general term for the Maclaurin series representing $\int_0^{1/2} f(t)dt$.

(d) Use the first three nonzero terms of the infinite series found in part (c) to approximate $\int_0^{1/2} f(t)dt$. What are the properties of the terms of the series representing $\int_0^{1/2} f(t)dt$ that guarantee that this approximation is within $\dfrac{1}{10{,}000}$ of the exact value of the integral?

36. (2007 BC6) Let *f* be the function given by $f(x)=e^{-x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use your answer to part (b) to find $\lim_{x\to 0}\frac{1-x^2-f(x)}{x^4}$.

(c) Write the first four nonzero terms of the Taylor series for $\int_0^x e^{-t^2}dt$ about $x = 0$. Use the first two terms of your answer to estimate $\int_0^{1/2} e^{-t^2}dt$.

(d) Explain why the estimate found in part (c) differs from the actual value of $\int_0^{1/2} e^{-t^2}dt$ by less than $\frac{1}{200}$.

37. (2007B BC6) Let *f* be the function given by $f(x)=6e^{-x/3}$ for all x.

(a) Find the first four nonzero terms and the general term for the Taylor series for *f* about $x = 0$.

(b) Let g be the function given by $g(x)=\int_0^x f(t)dt$. Find the first four nonzero terms and the general term for the Taylor series for g about $x = 0$.

(c) The function h satisfies $h(x)=kf'(ax)$ for all x, where a and k are constants. The Taylor series for h about $x = 0$ is given by $h(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$. Find the values of a and k.

38. (2008 BC3) calculator permited

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	8	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	31	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

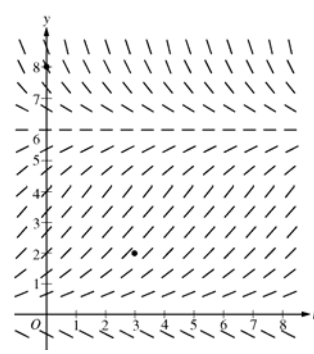
(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

39. (2008 BC6)

Consider the logistic differential equation $\frac{dy}{dt} = \frac{y}{8}(6 - y)$. Let $y = f(t)$ be the particular solution to the differential equation with $f(0) = 8$.

(a) A slope field for this differential equation is given below. Sketch possible solution curves through the points $(3, 2)$ and $(0, 8)$.
(Note: Use the axes provided in the exam booklet.)



(b) Use Euler's method, starting at $t = 0$ with two steps of equal size, to approximate $f(1)$.

(c) Write the second-degree Taylor polynomial for f about $t = 0$, and use it to approximate $f(1)$.

(d) What is the range of f for $t \ge 0$?

40. (2008B BC6)

Let f be the function given by $f(x) = \dfrac{2x}{1+x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.

(c) The derivative of $\ln\left(1+x^2\right)$ is $\dfrac{2x}{1+x^2}$. Write the first four nonzero terms of the Taylor series for $\ln\left(1+x^2\right)$ about $x = 0$.

(d) Use the series found in part (c) to find a rational number A such that $\left|A - \ln\left(\frac{5}{4}\right)\right| < \frac{1}{100}$. Justify your answer.

41. (2009 BC4)

Consider the differential equation $\dfrac{dy}{dx} = 6x^2 - x^2y$. Let $y = f(x)$ be a particular solution to this differential equation with the initial condition $f(-1) = 2$.

(a) Use Euler's method with two steps of equal size, starting at $x = -1$, to approximate $f(0)$. Show the work that leads to your answer.

(b) At the point $(-1, 2)$, the value of $\dfrac{d^2y}{dx^2}$ is -12. Find the second-degree Taylor polynomial for f about $x = -1$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.

42. (2009 BC6)

The Maclaurin series for e^x is $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \cdots + \dfrac{x^n}{n!} + \cdots$. The continuous function f is defined by $f(x) = \dfrac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function f has derivatives of all orders at $x = 1$.

(a) Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.

(b) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(c) Use the ratio test to find the interval of convergence for the Taylor series found in part (b).

(d) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

43. (2009B BC6)

Question 6

The function f is defined by the power series

$$f(x) = 1 + (x+1) + (x+1)^2 + \cdots + (x+1)^n + \cdots = \sum_{n=0}^{\infty} (x+1)^n$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The power series above is the Taylor series for f about $x = -1$. Find the sum of the series for f.

(c) Let g be the function defined by $g(x) = \int_{-1}^{x} f(t)\,dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ cannot be determined.

(d) Let h be the function defined by $h(x) = f\left(x^2 - 1\right)$. Find the first three nonzero terms and the general term of the Taylor series for h about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

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44. (2010 BC6)

$$f(x) = \begin{cases} \dfrac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\dfrac{1}{2} & \text{for } x = 0 \end{cases}$$

The function f, defined above, has derivatives of all orders. Let g be the function defined by $g(x) = 1 + \int_{0}^{x} f(t)\,dt$.

(a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.

(c) Write the fifth-degree Taylor polynomial for g about $x = 0$.

(d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x = 0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

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45. (2010B BC6)

Question 6

The Maclaurin series for the function f is given by $f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1}$ on its interval of convergence.

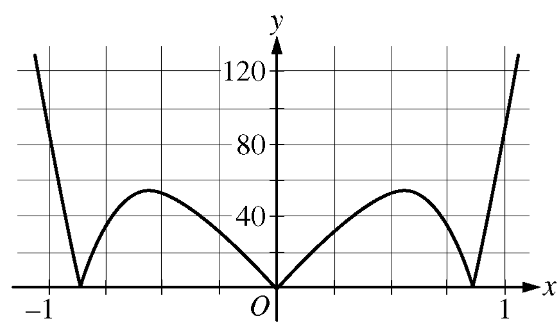
(a) Find the interval of convergence for the Maclaurin series of f. Justify your answer.

(b) Show that $y = f(x)$ is a solution to the differential equation $xy' - y = \frac{4x^2}{1+2x}$ for $|x| < R$, where R is the radius of convergence from part (a).

46. (2011 BC6)

Question 6

Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

47. (2011B BC6)

Let $f(x) = \ln\left(1 + x^3\right)$.

(a) The Maclaurin series for $\ln(1 + x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n} + \cdots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) The radius of convergence of the Maclaurin series for f is 1. Determine the interval of convergence. Show the work that leads to your answer.

(c) Write the first four nonzero terms of the Maclaurin series for $f'\left(t^2\right)$. If $g(x) = \int_0^x f'\left(t^2\right)\,dt$, use the first two nonzero terms of the Maclaurin series for g to approximate $g(1)$.

(d) The Maclaurin series for g, evaluated at $x = 1$, is a convergent alternating series with individual terms that decrease in absolute value to 0. Show that your approximation in part (c) must differ from $g(1)$ by less than $\frac{1}{5}$.

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48. (2012 BC6)

The function g has derivatives of all orders, and the Maclaurin series for g is

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots.$$

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

49. (2013 BC6)

A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

(a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

(c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

Answer Key

1.	A	1993	BC	#31	46%
2.	C	1985	BC	#14	82%
3.	A	1988	BC	#44	35%
4.	A	1993	BC	#16	57%
5.	B	1998	BC	#18	35%
6.	A	1998	BC	#22	68%
7.	D	1998	BC	#76	60%
8.	A	1985	BC	#10	49%
9.	E	1985	BC	#42	64%
10.	B	1988	BC	#13	77%
11.	A	1993	BC	#43	26%
12.	E	1998	BC	#14	68%
13.	D	1998	BC	#27	35%
14.	A	1998	BC	#89	56%
15.	D	1985	BC	#31	53%
16.	C	1988	BC	#38	52%
17.	C	1993	BC	#27	49%
18.	B	1998	BC	#84	40%