CALCULUS II

Solutions to Practice Problems
Series & Sequences

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Chapter 4 : Series & Sequences

Here is a listing of sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Sequences – In this section we define just what we mean by sequence in a math class and give the basic notation we will use with them. We will focus on the basic terminology, limits of sequences and convergence of sequences in this section. We will also give many of the basic facts and properties we'll need as we work with sequences.

More on Sequences – In this section we will continued examining sequences. We will determine if a sequence in an increasing sequence or a decreasing sequence and hence if it is a monotonic sequence. We will also determine a sequence is bounded below, bounded above and/or bounded.

Series – The Basics – In this section we will formally define an infinite series. We will also give many of the basic facts, properties and ways we can use to manipulate a series. We will also briefly discuss how to determine if an infinite series will converge or diverge (a more in depth discussion of this topic will occur in the next section).

Convergence/Divergence of Series – In this section we will discuss in greater detail the convergence and divergence of infinite series. We will illustrate how partial sums are used to determine if an infinite series converges or diverges. We will also give the Divergence Test for series in this section.

Special Series – In this section we will look at three series that either show up regularly or have some nice properties that we wish to discuss. We will examine Geometric Series, Telescoping Series, and Harmonic Series.

Integral Test – In this section we will discuss using the Integral Test to determine if an infinite series converges or diverges. The Integral Test can be used on a infinite series provided the terms of the series are positive and decreasing. A proof of the Integral Test is also given.

Comparison Test/Limit Comparison Test – In this section we will discuss using the Comparison Test and Limit Comparison Tests to determine if an infinite series converges or diverges. In order to use either test the terms of the infinite series must be positive. Proofs for both tests are also given.

Alternating Series Test – In this section we will discuss using the Alternating Series Test to determine if an infinite series converges or diverges. The Alternating Series Test can be used only if the terms of the series alternate in sign. A proof of the Alternating Series Test is also given.

Absolute Convergence – In this section we will have a brief discussion on absolute convergence and conditionally convergent and how they relate to convergence of infinite series.

Ratio Test – In this section we will discuss using the Ratio Test to determine if an infinite series converges absolutely or diverges. The Ratio Test can be used on any series, but unfortunately will not always yield a conclusive answer as to whether a series will converge absolutely or diverge. A proof of the Ratio Test is also given.

Root Test – In this section we will discuss using the Root Test to determine if an infinite series converges absolutely or diverges. The Root Test can be used on any series, but unfortunately will not always yield a conclusive answer as to whether a series will converge absolutely or diverge. A proof of the Root Test is also given.

Strategy for Series – In this section we give a general set of guidelines for determining which test to use in determining if an infinite series will converge or diverge. Note as well that there really isn't one set of guidelines that will always work and so you always need to be flexible in following this set of guidelines. A summary of all the various tests, as well as conditions that must be met to use them, we discussed in this chapter are also given in this section.

Estimating the Value of a Series – In this section we will discuss how the Integral Test, Comparison Test, Alternating Series Test and the Ratio Test can, on occasion, be used to estimating the value of an infinite series.

Power Series – In this section we will give the definition of the power series as well as the definition of the radius of convergence and interval of convergence for a power series. We will also illustrate how the Ratio Test and Root Test can be used to determine the radius and interval of convergence for a power series.

Power Series and Functions – In this section we discuss how the formula for a convergent Geometric Series can be used to represent some functions as power series. To use the Geometric Series formula, the function must be able to be put into a specific form, which is often impossible. However, use of this formula does quickly illustrate how functions can be represented as a power series. We also discuss differentiation and integration of power series.

Taylor Series – In this section we will discuss how to find the Taylor/Maclaurin Series for a function. This will work for a much wider variety of function than the method discussed in the previous section at the expense of some often unpleasant work. We also derive some well known formulas for Taylor series of $\mathbf{e}^x$, $\cos(x)$ and $\sin(x)$ around $x = 0$.

Applications of Series – In this section we will take a quick look at a couple of applications of series. We will illustrate how we can find a series representation for indefinite integrals that cannot be evaluated by any other method. We will also see how we can use the first few terms of a power series to approximate a function.

Binomial Series – In this section we will give the Binomial Theorem and illustrate how it can be used to quickly expand terms in the form $(a+b)^n$ when *n* is an integer. In addition, when *n* is not an integer an extension to the Binomial Theorem can be used to give a power series representation of the term.

Section 4-1 : Sequences

1. List the first 5 terms of the following sequence.

$$\left\{ \frac{4n}{n^2-7} \right\}_{n=0}^{\infty}$$

Solution

There really isn't all that much to this problem. All we need to do is, starting at $n=0$, plug in the first five values of *n* into the formula for the sequence terms. Doing that gives,

$$n=0: \qquad \frac{4(0)}{(0)^2-7}=0$$

$$n=1: \qquad \frac{4(1)}{(1)^2-7}=\frac{4}{-6}=-\frac{2}{3}$$

$$n=2: \qquad \frac{4(2)}{(2)^2-7}=\frac{8}{-3}=-\frac{8}{3}$$

$$n=3: \qquad \frac{4(3)}{(3)^2-7}=\frac{12}{2}=6$$

$$n=4: \qquad \frac{4(4)}{(4)^2-7}=\frac{16}{9}$$

So, the first five terms of the sequence are,

$$\boxed{\left\{ 0, -\frac{2}{3}, -\frac{8}{3}, 6, \frac{16}{9}, \ldots \right\}}$$

Note that we put the formal answer inside the braces to make sure that we don't forget that we are dealing with a sequence and we made sure and included the "…" at the end to reminder ourselves that there are more terms to this sequence that just the five that we listed out here.

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2. List the first 5 terms of the following sequence.

$$\left\{ \frac{(-1)^{n+1}}{2n+(-3)^n} \right\}_{n=2}^{\infty}$$

Solution

There really isn't all that much to this problem. All we need to do is, starting at $n = 2$, plug in the first five values of *n* into the formula for the sequence terms. Doing that gives,

$$n = 2: \qquad \frac{(-1)^{2+1}}{2(2)+(-3)^2} = \frac{-1}{13} = -\frac{1}{13}$$

$$n = 3: \qquad \frac{(-1)^{3+1}}{2(3)+(-3)^3} = \frac{1}{-21} = -\frac{1}{21}$$

$$n = 4: \qquad \frac{(-1)^{4+1}}{2(4)+(-3)^4} = \frac{-1}{89} = -\frac{1}{89}$$

$$n = 5: \qquad \frac{(-1)^{5+1}}{2(5)+(-3)^5} = \frac{1}{-233} = -\frac{1}{233}$$

$$n = 6: \qquad \frac{(-1)^{6+1}}{2(6)+(-3)^6} = \frac{-1}{741} = -\frac{1}{741}$$

So, the first five terms of the sequence are,

$$\boxed{\left\{-\frac{1}{13}, -\frac{1}{21}, -\frac{1}{89}, -\frac{1}{233}, -\frac{1}{741}, \ldots\right\}}$$

Note that we put the formal answer inside the braces to make sure that we don't forget that we are dealing with a sequence and we made sure and included the "…" at the end to reminder ourselves that there are more terms to this sequence that just the five that we listed out here.

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3. Determine if the given sequence converges or diverges. If it converges what is its limit?

$$\left\{\frac{n^2 - 7n + 3}{1 + 10n - 4n^2}\right\}_{n=3}^{\infty}$$

Step 1

To answer this all we need is the following limit of the sequence terms.

$$\lim_{n\to\infty} \frac{n^2 - 7n + 3}{1 + 10n - 4n^2} = -\frac{1}{4}$$

You do recall how to take limits at infinity right? If not you should go back into the Calculus I material do some refreshing on limits at infinity as well at L'Hospital's rule.

Step 2
We can see that the limit of the terms existed and was a finite number and so we know that the sequence **converges** and its limit is $-\frac{1}{4}$.

---

4. Determine if the given sequence converges or diverges. If it converges what is its limit?

$$\left\{ \frac{(-1)^{n-2} n^2}{4+n^3} \right\}_{n=0}^{\infty}$$

Step 1
To answer this all we need is the following limit of the sequence terms.

$$\lim_{n\to\infty} \frac{(-1)^{n-2} n^2}{4+n^3}$$

However, because of the $(-1)^{n-2}$ we can't compute this limit using our knowledge of computing limits from Calculus I.

Step 2
Recall however, that we had a nice Fact in the notes from this section that had us computing not the limit above but instead computing the limit of the absolute value of the sequence terms.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n-2} n^2}{4+n^3} \right| = \lim_{n\to\infty} \frac{n^2}{4+n^3} = 0$$

This is a limit that we can compute because the absolute value got rid of the alternating sign, *i.e.* the $(-1)^{n+2}$.

Step 3
Now, by the Fact from class we know that because the limit of the absolute value of the sequence terms was zero (and recall that to use that fact the limit MUST be zero!) we also know the following limit.

$$\lim_{n\to\infty} \frac{(-1)^{n-2} n^2}{4+n^3} = 0$$

Step 4
We can see that the limit of the terms existed and was a finite number and so we know that the sequence **converges** and its limit is zero.

---

5. Determine if the given sequence converges or diverges. If it converges what is its limit?

$$\left\{ \frac{\mathbf{e}^{5n}}{3-\mathbf{e}^{2n}} \right\}_{n=1}^{\infty}$$

Step 1
To answer this all we need is the following limit of the sequence terms.

$$\lim_{n\to\infty} \frac{\mathbf{e}^{5n}}{3-\mathbf{e}^{2n}} = \lim_{n\to\infty} \frac{5\mathbf{e}^{5n}}{-2\mathbf{e}^{2n}} = \lim_{n\to\infty} \frac{5}{-2}\mathbf{e}^{3n} = -\infty$$

You do recall how to use L'Hospital's rule to compute limits at infinity right? If not you should go back into the Calculus I material do some refreshing.

Step 2
We can see that the limit of the terms existed and but was infinite and so we know that the sequence **diverges**.

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6. Determine if the given sequence converges or diverges. If it converges what is its limit?

$$\left\{ \frac{\ln(n+2)}{\ln(1+4n)} \right\}_{n=1}^{\infty}$$

Step 1
To answer this all we need is the following limit of the sequence terms.

$$\lim_{n\to\infty} \frac{\ln(n+2)}{\ln(1+4n)} = \lim_{n\to\infty} \frac{1/_{n+2}}{4/_{1+4n}} = \lim_{n\to\infty} \frac{1+4n}{4(n+2)} = 1$$

You do recall how to use L'Hospital's rule to compute limits at infinity right? If not, you should go back into the Calculus I material do some refreshing.

Step 2
We can see that the limit of the terms existed and was a finite number and so we know that the sequence **converges** and its limit is one.

Section 4-2 : More on Sequences

1. Determine if the following sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

$$\left\{\frac{1}{4n}\right\}_{n=1}^{\infty}$$

Hint : There is no one set process for finding all this information. Sometimes it is easier to find one set of information before the other and at other times it doesn't matter which set of information you find first. This is one of those sequences that it doesn't matter which set of information you find first and both sets should be fairly easy to determine the answers without a lot of work.

Step 1
For this problem let's get the bounded information first as that seems to be pretty simple.

First note that because both the numerator and denominator are positive then the quotient is also positive and so we can see that the sequence must be **bounded below by zero**.

Next let's note that because we are starting with $n=1$ the denominator will always be $4n \ge 4 > 1$ and so we can also see that the sequence must be **bounded above by one**. Note that, in this case, this not the "best" upper bound for the sequence but the problem didn't ask for that. For this sequence we'll be able to get a better one once we have the increasing/decreasing information.

Because the sequence is bounded above and bounded below the sequence is also **bounded**.

Step 2
For the increasing/decreasing information we can see that, for our range of $n \ge 1$, we have,

$$4n < 4\left(n+1\right)$$

and so,

$$\frac{1}{4n} > \frac{1}{4\left(n+1\right)}$$

If we define $a_n = \dfrac{1}{4n}$ this in turn tells us that $a_n > a_{n+1}$ for all $n \ge 1$ and so the sequence is **decreasing** and hence **monotonic**.

Note that because we have now determined that the sequence is decreasing we can see that the "best" upper bound would be the first term of the sequence or, $\frac{1}{4}$.

2. Determine if the following sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

$$\left\{ n\left(-1\right)^{n+2} \right\}_{n=0}^{\infty}$$

Hint : There is no one set process for finding all this information. Sometimes it is easier to find one set of information before the other and at other times it doesn't matter which set of information you find first. This is one of those sequences that it doesn't matter which set of information you find first and both sets should be fairly easy to determine the answers without a lot of work.

Step 1
For this problem let's get the increasing/decreasing information first as that seems to be pretty simple and will help at least a little bit with the bounded information.

In this case let's just write out the first few terms of the sequence.

$$\left\{ n\left(-1\right)^{n+2} \right\}_{n=0}^{\infty} = \left\{ 0,\ -1,\ 2,\ -3,\ 4,\ -5,\ 6,\ -7,\ldots \right\}$$

Just from the first three terms we can see that this sequence is **not an increasing sequence** and it is **not a decreasing sequence** and therefore is **not monotonic**.

Step 2
Now let's see what bounded information we can get.

From the first few terms of the sequence we listed out above we can see that each successive term will get larger and change signs. Therefore, there cannot be an upper or a lower bound for the sequence. No matter what value we would try to use for an upper or a lower bound all we would need to do is take *n* large enough and we would eventually get a sequence term that would go past the proposed bound.

Therefore, this sequence is **not bounded**.

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3. Determine if the following sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

$$\left\{ 3^{-n} \right\}_{n=0}^{\infty}$$

Hint : There is no one set process for finding all this information. Sometimes it is easier to find one set of information before the other and at other times it doesn't matter which set of information you find first. For this sequence it might be a little easier to find the bounds (if any exist) if you first have the increasing/decreasing information.

Step 1

For this problem let's get the increasing/decreasing information first as that seems to be pretty simple and will help at least a little bit with the bounded information.

We' all agree that, for our range of $n \geq 0$, we have,

$$n < n+1$$

This in turn gives,

$$3^{-n} = \frac{1}{3^n} > \frac{1}{3^{n+1}} = 3^{-(n+1)}$$

So, if we define $a_n = 3^{-n}$ we have $a_n > a_{n+1}$ for all $n \geq 0$ and so the sequence is **decreasing** and hence is also **monotonic**.

Step 2
Now let's see what bounded information we can get.

First, it is hopefully obvious that all the terms are positive and so the sequence is **bounded below by zero**.

Next, we saw in the first step that the sequence was decreasing and so the first term will be the largest term and so the sequence is **bounded above by** $3^{-(0)} = 1$ (*i.e.* the $n = 0$ sequence term).

Therefore, because this sequence is bounded below and bounded above the sequence is **bounded**.

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4. Determine if the following sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

$$\left\{ \frac{2n^2 - 1}{n} \right\}_{n=2}^{\infty}$$

Hint : There is no one set process for finding all this information. Sometimes it is easier to find one set of information before the other and at other times it doesn't matter which set of information you find first. For this sequence having the increasing/decreasing information will probably make the determining the bounds (if any exist) somewhat easier.

Step 1
For this problem let's get the increasing/decreasing information first.

For Problems 1 – 3 in this section it was easy enough to just ask what happens if we increase *n* to determine the increasing/decreasing information for this problem. However, in this case, increasing *n* will increase both the numerator and denominator and so it would be somewhat more difficult to do that analysis here.

Therefore, we will resort to some quick Calculus I to determine increasing/decreasing information. We can define the following function and take its derivative.

$$f\left(x\right)=\frac{2x^2-1}{x} \qquad \Rightarrow \qquad f'\left(x\right)=\frac{2x^2+1}{x^2}$$

We can clearly see that the derivative will always be positive for $x \neq 0$ and so the function is increasing for $x \neq 0$. Therefore, because the function values are the same as the sequence values when *x* is an integer we can see that the sequence, which starts at $n = 2$, must also be **increasing** and hence it is also **monotonic**.

Step 2
Now let's see what bounded information we can get.

First, it is hopefully obvious that all the terms are positive for our range of $n \geq 2$ and so the sequence is **bounded below by zero**. We could also use the fact that the sequence is increasing the first term would have to be the smallest term in the sequence and so a better lower bound would be the first sequence term which is $\frac{7}{2}$. Either would work for this problem.

Now let's see what we can determine about an upper bound (provided it has one of course…).

We know that the function is increasing but that doesn't mean there is no upper bound. Take a look at Problems 1 and 3 above. Each of those were decreasing sequences and yet they had a lower bound. Do not make the mistake of assuming that an increasing sequence will not have an upper bound or a decreasing sequence will not have a lower bound. Sometimes they will and sometimes they won't.

For this sequence we'll need to approach any potential upper bound a little differently than the previous problems. Let's first compute the following limit of the terms,

$$\lim_{n\to\infty}\frac{2n^2-1}{n}=\lim_{n\to\infty}\left(2n-\frac{1}{n}\right)=\infty$$

Since the limit of the terms is infinity we can see that the terms will increase without bound. Therefore, in this case, there really is **no upper bound** for this sequence. Please remember the warning above however! Just because this increasing sequence had no upper bound does not mean that every increasing sequence will have an upper bound.

Finally, because this sequence is bounded below but not bounded above the sequence is **not bounded**.

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5. Determine if the following sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

$$\left\{\frac{4-n}{2n+3}\right\}_{n=1}^{\infty}$$

Hint : There is no one set process for finding all this information. Sometimes it is easier to find one set of information before the other and at other times it doesn't matter which set of information you find first. For this sequence having the increasing/decreasing information will probably make the determining the bounds (if any exist) somewhat easier.

Step 1
For this problem let's get the increasing/decreasing information first.

For Problems 1 – 3 in this section it was easy enough to just ask what happens if we increase *n* to determine the increasing/decreasing information for this problem. However, in this case, increasing *n* will increase both the numerator and denominator and so it would be somewhat more difficult to do that analysis here.

Therefore, we will resort to some quick Calculus I to determine increasing/decreasing information. We can define the following function and take its derivative.

$$f\left(x\right)=\frac{4-x}{2x+3} \qquad \Rightarrow \qquad f'\left(x\right)=\frac{-11}{\left(2x+3\right)^2}$$

We can clearly see that the derivative will always be negative for $x \neq -\frac{3}{2}$ and so the function is decreasing for $x \neq -\frac{3}{2}$. Therefore, because the function values are the same as the sequence values when *x* is an integer we can see that the sequence, which starts at $n=1$, must also be **decreasing** and hence it is also **monotonic**.

Step 2
Now let's see what bounded information we can get.

First, because the sequence is decreasing we can see that the first term of the sequence will be the largest and hence will also be an upper bound for the sequence. So, the sequence is **bounded above** by $\frac{3}{5}$ (*i.e.* the $n=1$ sequence term).

Next let's look for the lower bound (if it exists). For this problem let's first take a quick look at the limit of the sequence terms. In this case the limit of the sequence terms is,

$$\lim_{n\to\infty}\frac{4-n}{2n+3}=-\frac{1}{2}$$

Recall what this limit tells us about the behavior of our sequence terms. The limit means that as $n\to\infty$ the sequence terms must be getting closer and closer to $-\frac{1}{2}$.

Now, for a second, let's suppose that that $-\frac{1}{2}$ is not a lower bound for the sequence terms and let's also keep in mind that we've already determined that the sequence is decreasing (means that each successive term must be smaller than (*i.e.* below) the previous one…).

So, if $-\frac{1}{2}$ is not a lower bound then we know that somewhere there must be sequence terms below (or smaller than) $-\frac{1}{2}$. However, because we also know that terms must be getting closer and closer to $-\frac{1}{2}$ and we've now assumed there are terms below $-\frac{1}{2}$ the only way for that to happen at this point is for at least a few sequence terms to increase up towards $-\frac{1}{2}$ (remember we've assumed there are terms below this!). That can't happen however because we know the sequence is a decreasing sequence.

Okay, what was the point of all this? Well recall that we got to this apparent contradiction to the decreasing nature of the sequence by first assuming that $-\frac{1}{2}$ was not a lower bound. Since making this assumption led us to something that can't possibly be true that in turn means that $-\frac{1}{2}$ must in fact be a lower bound since we've shown that sequence terms simply cannot go below this value!

Therefore, the sequence is **bounded below** by $-\frac{1}{2}$.

Finally, because this sequence is both bounded above and bounded below the sequence is **bounded**.

Before leaving this problem a quick word of caution. The limit of a sequence is not guaranteed to be a bound (upper or lower) for a sequence. It will only be a bound under certain circumstances and so we can't simply compute the limit and assume it will be a bound for every sequence! Can you see a condition that will allow the limit to be a bound?

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6. Determine if the following sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

$$\left\{\frac{-n}{n^2+25}\right\}_{n=2}^{\infty}$$

Hint : There is no one set process for finding all this information. Sometimes it is easier to find one set of information before the other and at other times it doesn't matter which set of information you find first. For this sequence having the increasing/decreasing information will probably make the determining the bounds (if any exist) somewhat easier.

Step 1
For this problem let's get the increasing/decreasing information first.

For Problems 1 – 3 in this section it was easy enough to just ask what happens if we increase *n* to determine the increasing/decreasing information for this problem. However, in this case, increasing *n* will increase both the numerator and denominator and so it would be somewhat more difficult to do that analysis here.

Therefore, we will resort to some quick Calculus I to determine increasing/decreasing information. We can define the following function and take its derivative.

$$f\left(x\right)=\frac{-x}{x^2+25} \qquad \Rightarrow \qquad f'\left(x\right)=\frac{x^2-25}{\left(x^2+25\right)^2}$$

Hopefully, it's fairly clear that the critical points of the function are $x=\pm 5$. We'll leave it to you to draw a quick number line or sign chart to verify that the function will be decreasing in the range $2 \le x < 5$ and increasing in the range $x > 5$. Note that we just looked at the ranges of *x* that correspond to the ranges of *n* for our sequence here.

Now, because the function values are the same as the sequence values when *x* is an integer we can see that the sequence, which starts at $n=2$, has terms that increase and terms that decrease and hence the sequence is **not an increasing sequence** and the sequence is **not a decreasing sequence**. That also means that the sequence is **not monotonic**.

Step 2
Now let's see what bounded information we can get.

In this case, unlike many of the previous problems in this section, we don't have a monotonic sequence. However, we can still use the increasing/decreasing information above to help us out with the bounds.

First, we know that the sequence is decreasing in the range $2 \le n < 5$ and increasing in the range $n > 5$. From our Calculus I knowledge we know that this means $n=5$ must be a minimum of the sequence terms and hence the sequence is **bounded below** by $\frac{-5}{50}=-\frac{1}{10}$ (*i.e.* the $n=5$ sequence term).

Next let's look for the upper bound (if it exists). For this problem let's first take a quick look at the limit of the sequence terms. In this case the limit of the sequence terms is,

$$\lim_{n\to\infty}\frac{-n}{n^2+25}=0$$

Recall what this limit tells us about the behavior of our sequence terms. The limit means that as $n \to \infty$ the sequence terms must be getting closer and closer to zero.

Now, for a second, let's look at just the portion of the sequence with $n > 5$ and let's further suppose that zero is not an upper bound for the sequence terms with $n > 5$. Let's also keep in mind that we've already determined that the sequence is increasing for $n > 5$ (means that each successive term must be larger than (*i.e.* above) the previous one…).

So, if zero is not an upper bound (for $n > 5$) then we know that somewhere there must be sequence terms with $n > 5$ above (or larger than) zero. So, we know that terms must be getting closer and closer to zero and we've now assumed there are terms above zero. Therefore the only way for the terms to approach the limit of zero is for at least a few sequence terms with $n > 5$ to decrease down towards zero (remember we've assumed there are terms above this!). That can't happen however because we know that for $n > 5$ the sequence is increasing.

Okay, what was the point of all this? Well recall that we got to this apparent contradiction to the increasing nature of the sequence for $n > 5$ by first assuming that zero was not an upper bound for the portion of the sequence with $n > 5$. Since making this assumption led us to something that can't possibly be true that in turn means that zero must in fact be an upper bound for the portion of the sequence with $n > 5$ since we've shown that sequence terms simply cannot go above this value!

Note that we've not yet actually shown that zero in an upper bound for the sequence and in fact it might not actually be an upper bound. However, what we have shown is that it is an upper bound for the vast majority of the sequence, *i.e.* for the portion of the sequence with $n > 5$.

All we need to do to finish the upper bound portion of this problem off is check what the first few terms of the sequence are doing. There are several ways to do this. One is to just compute the remaining initial terms of the sequence to see if they are above or below zero. For this sequence that isn't too bad as there are only 4 terms ($n = 2, 3, 4, 5$). However, if there'd been several hundred terms that wouldn't be so easy so let's take a look at another approach that will always be easy to do in this case because we have the increasing/decreasing information for this initial portion of the sequence.

Let's simply note that for the first part of this sequence we've already shown that the sequence is decreasing. Therefore, the very first sequence term of $-\frac{2}{29}$ (*i.e.* the $n = 2$ sequence term) will be the largest term for this initial bit of the sequence that is decreasing. This term is clearly less than zero and so zero will also be larger than all the remaining terms in the initial decreasing portion of the sequence and hence the sequence is **bounded above by zero**.

Finally, because this sequence is both bounded above and bounded below the sequence is **bounded**.

Before leaving this problem a couple of quick words of caution.

First, the limit of a sequence is not guaranteed to be a bound (upper or lower) for a sequence so be careful to not just always assume that the limit is an upper/lower bound for a sequence.

Second, as this problem has shown determining the bounds of a sequence can sometimes be a fairly involved process that involves quite a bit of work and lots of various pieces of knowledge about the other behavior of the sequence.

Section 4-3 : Series - The Basics

1. Perform an index shift so that the following series starts at $n = 3$.

$$\sum_{n=1}^{\infty} \left(n 2^{n} - 3^{1-n} \right)$$

Solution

There really isn't all that much to this problem. Just remember that, in this case, we'll need to increase the initial value of the index by two so it will start at $n = 3$ and this means all the *n*'s in the series terms will need to decrease by the same amount (two in this case…).

Doing this gives the following series.

$$\sum_{n=1}^{\infty} \left(n 2^{n} - 3^{1-n} \right) = \sum_{n=3}^{\infty} \left(\left(n-2 \right) 2^{n-2} - 3^{1-(n-2)} \right) = \boxed{\sum_{n=3}^{\infty} \left(\left(n-2 \right) 2^{n-2} - 3^{3-n} \right)}$$

Be careful with parenthesis, exponents, coefficients and negative signs when "shifting" the *n*'s in the series terms. When replacing *n* with $n - 2$ make sure to add in parenthesis where needed to preserve coefficients and minus signs.

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2. Perform an index shift so that the following series starts at $n = 3$.

$$\sum_{n=7}^{\infty} \frac{4-n}{n^{2}+1}$$

Solution

There really isn't all that much to this problem. Just remember that, in this case, we'll need to decrease the initial value of the index by four so it will start at $n = 3$ and this means all the *n*'s in the series terms will need to increase by the same amount (four in this case…).

Doing this gives the following series.

$$\sum_{n=7}^{\infty} \frac{4-n}{n^{2}+1} = \sum_{n=3}^{\infty} \frac{4-\left(n+4 \right)}{\left(n+4 \right)^{2}+1} = \boxed{\sum_{n=3}^{\infty} \frac{-n}{\left(n+4 \right)^{2}+1}}$$

Be careful with parenthesis, exponents, coefficients and negative signs when "shifting" the *n*'s in the series terms. When replacing *n* with $n + 4$ make sure to add in parenthesis where needed to preserve coefficients and minus signs.

---

3. Perform an index shift so that the following series starts at $n = 3$.

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-3}(n+2)}{5^{1+2n}}$$

Solution

There really isn't all that much to this problem. Just remember that, in this case, we'll need to increase the initial value of the index by one so it will start at $n = 3$ and this means all the *n*'s in the series terms will need to decrease by the same amount (one in this case…).

Doing this gives the following series.

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-3}(n+2)}{5^{1+2n}} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1-3}(n-1+2)}{5^{1+2(n-1)}} = \boxed{\sum_{n=3}^{\infty} \frac{(-1)^{n-4}(n+1)}{5^{2n-1}}}$$

Be careful with parenthesis, exponents, coefficients and negative signs when "shifting" the *n*'s in the series terms. When replacing *n* with $n-1$ make sure to add in parenthesis where needed to preserve coefficients and minus signs.

---

4. Strip out the first 3 terms from the series $\sum_{n=1}^{\infty} \frac{2^{-n}}{n^2+1}$.

Solution

Remember that when we say we are going to "strip out" terms from a series we aren't really getting rid of them. All we are doing is writing the first few terms of the series as a summation in front of the series.

So, for this series stripping out the first three terms gives,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2^{-n}}{n^2+1} &= \frac{2^{-1}}{1^2+1} + \frac{2^{-2}}{2^2+1} + \frac{2^{-3}}{3^2+1} + \sum_{n=4}^{\infty} \frac{2^{-n}}{n^2+1} \\ &= \frac{1}{4} + \frac{1}{20} + \frac{1}{80} + \sum_{n=4}^{\infty} \frac{2^{-n}}{n^2+1} \\ &= \boxed{\frac{5}{16} + \sum_{n=4}^{\infty} \frac{2^{-n}}{n^2+1}}\end{aligned}$$

This first step isn't really all that necessary but was included here to make it clear that we were plugging in $n=1$, $n=2$ and $n=3$ (*i.e.* the first three values of *n*) into the general series term. Also, don't forget to change the starting value of *n* to reflect the fact that we've "stripped out" the first three values of *n* or terms.

---

5. Given that $\sum_{n=0}^{\infty} \frac{1}{n^3+1} = 1.6865$ determine the value of $\sum_{n=2}^{\infty} \frac{1}{n^3+1}$.

Step 1
First notice that if we strip out the first two terms from the series that starts at $n=0$ the result will involve a series that starts at $n=2$.

Doing this gives,

$$\sum_{n=0}^{\infty} \frac{1}{n^3+1} = \frac{1}{0^3+1} + \frac{1}{1^3+1} + \sum_{n=2}^{\infty} \frac{1}{n^3+1} = \frac{3}{2} + \sum_{n=2}^{\infty} \frac{1}{n^3+1}$$

Step 2
Now, for this situation we are given the value of the series that starts at $n=0$ and are asked to determine the value of the series that starts at $n=2$. To do this all we need to do is plug in the known value of the series that starts at $n=0$ into the "equation" above and "solve" for the value of the series that starts at $n=2$.

This gives,

$$1.6865 = \frac{3}{2} + \sum_{n=2}^{\infty} \frac{1}{n^3+1} \qquad \Rightarrow \qquad \sum_{n=2}^{\infty} \frac{1}{n^3+1} = 1.6865 - \frac{3}{2} = \boxed{0.1865}$$

Section 4-4 : Convergence/Divergence of Series

1. Compute the first 3 terms in the sequence of partial sums for the following series.

$$\sum_{n=1}^{\infty} n\,2^n$$

Solution

Remember that n^{th} term in the sequence of partial sums is just the sum of the first n terms of the series. So, computing the first three terms in the sequence of partial sums is pretty simple to do.

Here is the work for this problem.

$$\begin{aligned} s_1 &= (1)2^1 = 2 \\ s_2 &= (1)2^1 + (2)2^2 = 10 \\ s_3 &= (1)2^1 + (2)2^2 + (3)2^3 = 34 \end{aligned}$$

---

2. Compute the first 3 terms in the sequence of partial sums for the following series.

$$\sum_{n=3}^{\infty} \frac{2n}{n+2}$$

Solution

Remember that n^{th} term in the sequence of partial sums is just the sum of the first n terms of the series. So, computing the first three terms in the sequence of partial sums is pretty simple to do.

Here is the work for this problem.

$$\begin{aligned} s_3 &= \frac{2(3)}{3+2} = \frac{6}{5} \\ s_4 &= \frac{2(3)}{3+2} + \frac{2(4)}{4+2} = \frac{38}{15} \\ s_5 &= \frac{2(3)}{3+2} + \frac{2(4)}{4+2} + \frac{2(5)}{5+2} = \frac{416}{105} \end{aligned}$$

---

3. Assume that the n^{th} term in the sequence of partial sums for the series $\sum_{n=0}^{\infty} a_n$ is given below. Determine if the series $\sum_{n=0}^{\infty} a_n$ is convergent or divergent. If the series is convergent determine the value of the series.

$$s_n = \frac{5+8n^2}{2-7n^2}$$

Solution
There really isn't all that much that we need to do here other than to recall,

$$\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} s_n$$

So, to determine if the series converges or diverges, all we need to do is compute the limit of the sequence of the partial sums. The limit of the sequence of partial sums is,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{5+8n^2}{2-7n^2} = -\frac{8}{7}$$

Now, we can see that this limit exists and is finite (*i.e.* is not plus/minus infinity). Therefore, we now know that the series, $\sum_{n=0}^{\infty} a_n$, **converges** and its value is,

$$\sum_{n=0}^{\infty} a_n = -\frac{8}{7}$$

If you are unfamiliar with limits at infinity then you really need to go back to the Calculus I material and do some review of limits at infinity and L'Hospital's Rule as we will be doing quite a bit of these kinds of limits off and on over the next few sections.

---

4. Assume that the n^{th} term in the sequence of partial sums for the series $\sum_{n=0}^{\infty} a_n$ is given below. Determine if the series $\sum_{n=0}^{\infty} a_n$ is convergent or divergent. If the series is convergent determine the value of the series.

$$s_n = \frac{n^2}{5+2n}$$

Solution
There really isn't all that much that we need to do here other than to recall,

$$\sum_{n=0}^{\infty} a_n = \lim_{n \to \infty} s_n$$

So, to determine if the series converges or diverges, all we need to do is compute the limit of the sequence of the partial sums. The limit of the sequence of partial sums is,

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{n^2}{5+2n} = \infty$$

Now, we can see that this limit exists and but is infinite. Therefore, we now know that the series, $\sum_{n=0}^{\infty} a_n$, **diverges**.

If you are unfamiliar with limits at infinity then you really need to go back to the Calculus I material and do some review of limits at infinity and L'Hospital's Rule as we will be doing quite a bit of these kinds of limits off and on over the next few sections.

---

5. Show that the following series is divergent.

$$\sum_{n=0}^{\infty} \frac{3n\,\mathbf{e}^n}{n^2+1}$$

Solution
First let's note that we're being asked to show that the series is divergent. We are not being asked to determine **if** the series is divergent. At this point we really only know of two ways to actually show this.

The first option is to show that the limit of the sequence of partial sums either doesn't exist or is infinite. The problem with this approach is that for many series determining the general formula for the n^{th} term of the sequence of partial sums is very difficult if not outright impossible to do. That is true for this series and so that is not really a viable option for this problem.

Luckily enough for us there is actually an easier option to simply show that a series is divergent. All we need to do is use the Divergence Test.

The limit of the series terms is,

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{3n\,\mathbf{e}^n}{n^2+1} = \infty \neq 0$$

The limit of the series terms is not zero and so by the Divergence Test we know that the series in this problem is **divergence**.

---

6. Show that the following series is divergent.

$$\sum_{n=5}^{\infty}\frac{6+8n+9n^2}{3+2n+n^2}$$

Solution

First let's note that we're being asked to show that the series is divergent. We are not being asked to determine **if** the series is divergent. At this point we really only know of two ways to actually show this.

The first option is to show that the limit of the sequence of partial sums either doesn't exist or is infinite. The problem with this approach is that for many series determining the general formula for the n^{th} term of the sequence of partial sums is very difficult if not outright impossible to do. That is true for this series and so that is not really a viable option for this problem.

Luckily enough for us there is actually an easier option to simply show that a series is divergent. All we need to do is use the Divergence Test.

The limit of the series terms is,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\frac{6+8n+9n^2}{3+2n+n^2} = 9 \neq 0$$

The limit of the series terms is not zero and so by the Divergence Test we know that the series in this problem is **divergence**.

Section 4-5 : Special Series

1. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=0}^{\infty} 3^{2+n}\, 2^{1-3n}$$

Step 1
Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is a geometric series.

Step 2
Let's also notice that the initial value of the index is $n = 0$ and so we can put this into the form,

$$\sum_{n=0}^{\infty} a\, r^n$$

At that point we'll be able to determine if it converges or diverges and the value of the series if it does happen to converge.

In this case it's pretty simple to put the series into the form above so here is that work.

$$\sum_{n=0}^{\infty} 3^{2+n}\, 2^{1-3n} = \sum_{n=0}^{\infty} 3^2 3^n\, 2^1 2^{-3n} = \sum_{n=0}^{\infty} (9)(2)\frac{3^n}{2^{3n}} = \sum_{n=0}^{\infty} 18\frac{3^n}{8^n} = \sum_{n=0}^{\infty} 18\left(\frac{3}{8}\right)^n$$

Make sure you properly deal with any negative exponents that might happen to be in the terms!

Also recall that all the exponents must be simply *n* and can't be 3*n* or anything else. So, for this problem, we'll need to use basic exponent rules to write $2^{3n} = \left(2^3\right)^n = 8^n$.

Step 3
With the series in "proper" form we can see that $a = 18$ and $r = \frac{3}{8}$. Therefore, because we can clearly see that $\left|r\right| = \frac{3}{8} < 1$, the series will **converge** and its value is,

$$\sum_{n=0}^{\infty} 3^{2+n}\, 2^{1-3n} = \sum_{n=0}^{\infty} 18\left(\frac{3}{8}\right)^n = \frac{18}{1-\frac{3}{8}} = \boxed{\frac{144}{5}}$$

---

2. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=1}^{\infty} \frac{5}{6n}$$

Step 1
Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is a harmonic series.

Step 2
So because this is a harmonic series we know that it will **diverge**.

---

3. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=1}^{\infty} \frac{\left(-6\right)^{3-n}}{8^{2-n}}$$

Step 1
Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is a geometric series.

Step 2
Let's also notice that the initial value of the index is $n = 1$ and so we can put this into the form,

$$\sum_{n=1}^{\infty} a\, r^{n-1}$$

At that point we'll be able to determine if it converges or diverges and the value of the series if it does happen to converge.

So, let's get started on the work to put the series into the form above. First, let's get take care of the fact that both the *n*'s in the exponents are negative and they should be positive. Converting to positive *n*'s gives,

$$\sum_{n=1}^{\infty} \frac{\left(-6\right)^{3-n}}{8^{2-n}} = \sum_{n=1}^{\infty} \frac{8^{n-2}}{\left(-6\right)^{n-3}}$$

Note that how you chose to deal with the 3 and the 2 in the respective exponents is up to you. You can either do it the way we did here or strip them out and then move the terms to the numerator or denominator.

As noted above we need the two exponents to be $n-1$. This is an easy "fix" if we note that using basic exponent properties we can write each term as follows,

$$8^{n-2} = 8^{n-1}8^{-1} \qquad\qquad \left(-6\right)^{n-3} = \left(-6\right)^{n-1}\left(-6\right)^{-2}$$

With these two rewrites the series becomes,

$$\sum_{n=1}^{\infty} \frac{(-6)^{3-n}}{8^{2-n}} = \sum_{n=1}^{\infty} \frac{8^{n-1}8^{-1}}{(-6)^{n-1}(-6)^{-2}} = \sum_{n=1}^{\infty} \frac{(-6)^2}{8^1}\left(\frac{8}{-6}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{9}{2}\left(-\frac{4}{3}\right)^{n-1}$$

Step 3
With the series in "proper" form we can see that $a = \frac{9}{2}$ and $r = -\frac{4}{3}$. Therefore, because we can clearly see that $|r| = \left|-\frac{4}{3}\right| = \frac{4}{3} > 1$, the series will **diverge.**

---

4. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=1}^{\infty} \frac{3}{n^2 + 7n + 12}$$

Step 1
Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is not a geometric or harmonic series. That only leaves telescoping as a possibility.

Step 2
Now, we need to be careful here. There is no way to actually identify the series as a telescoping series at this point. We are only hoping that it is a telescoping series.

Therefore, the first real step here is to perform partial fractions on the series term to see what we get. Here is the partial fraction work for the series term.

$$\frac{3}{n^2 + 7n + 12} = \frac{3}{(n+3)(n+4)} = \frac{A}{n+3} + \frac{B}{n+4} \qquad \to \qquad 3 = A(n+4) + B(n+3)$$

$$\begin{array}{ll} n = -3 & 3 = A \\ n = -4 & 3 = -B \end{array} \qquad \to \qquad \begin{array}{l} A = 3 \\ B = -3 \end{array}$$

The series term in partial fraction form is then,

$$\frac{3}{n^2 + 7n + 12} = \frac{3}{n+3} - \frac{3}{n+4}$$

Step 3
The partial sums for this series are then,

$$s_n = \sum_{i=1}^{n} \left[\frac{3}{i+3} - \frac{3}{i+4}\right]$$

Step 4
Expanding the partial sums from the previous step give,

$$s_n = \sum_{i=1}^{n}\left[\frac{3}{i+3}-\frac{3}{i+4}\right] = \left[\frac{3}{4}-\frac{3}{5}\right]+\left[\frac{3}{5}-\frac{3}{6}\right]+\left[\frac{3}{6}-\frac{3}{7}\right]+\cdots$$
$$+\left[\frac{3}{n+1}-\frac{3}{n+2}\right]+\left[\frac{3}{n+2}-\frac{3}{n+3}\right]+\left[\frac{3}{n+3}-\frac{3}{n+4}\right]$$
$$=\frac{3}{4}-\frac{3}{n+4}$$

It is important when doing this expanding to expand out from both the initial and final values of *i* and to expand out until all the parts of a series term cancel. Once that has been done it is safe to assume that the cancelling will continue until we get near the end of the expansion.

Note that at this point we now know that the series was a telescoping series since we got all the "interior" terms to cancel out.

Step 5
At this point all we need to do is look at the limit of the partial sums to get,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left[\frac{3}{4}-\frac{3}{n+4}\right] = \frac{3}{4}$$

Step 6
The limit of the partial sums exists and is a finite number (*i.e.* not infinity) and so we can see that the series **converges** and its value is,

$$\sum_{n=1}^{\infty}\frac{3}{n^2+7n+12} = \frac{3}{4}$$

---

5. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=1}^{\infty}\frac{5^{n+1}}{7^{n-2}}$$

Step 1
Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is a geometric series.

Step 2
Let's also notice that the initial value of the index is $n=1$ and so we can put this into the form,

$$\sum_{n=1}^{\infty} a\,r^{n-1}$$

At that point we'll be able to determine if it converges or diverges and the value of the series if it does happen to converge.

As noted above we need the two exponents to be $n-1$. This is an easy "fix" if we note that using basic exponent properties we can write each term as follows,

$$5^{n+1} = 5^{n-1}5^{2} \qquad\qquad 7^{n-2} = 7^{n-1}7^{-1}$$

With these two rewrites the series becomes,

$$\sum_{n=1}^{\infty}\frac{5^{n+1}}{7^{n-2}} = \sum_{n=1}^{\infty}\frac{5^{n-1}5^{2}}{7^{n-1}7^{-1}} = \sum_{n=1}^{\infty}(25)(7)\frac{5^{n-1}}{7^{n-1}} = \sum_{n=1}^{\infty}175\left(\frac{5}{7}\right)^{n-1}$$

Step 3
With the series in "proper" form we can see that $a = 175$ and $r = \frac{5}{7}$. Therefore, because we can clearly see that $\left|r\right| = \frac{5}{7} < 1$, the series will **converge** and its value is,

$$\sum_{n=1}^{\infty}\frac{5^{n+1}}{7^{n-2}} = \sum_{n=1}^{\infty}175\left(\frac{5}{7}\right)^{n-1} = \frac{175}{1-\frac{5}{7}} = \boxed{\frac{1225}{2}}$$

---

6. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=2}^{\infty}\frac{5^{n+1}}{7^{n-2}}$$

Step 1
Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is a geometric series.

Step 2
Now, while we have correctly identified this as a geometric series it doesn't start at either of the two standard starting values of *n*, *i.e.* $n = 0$ or $n = 1$.

This won't stop us from determining if the series converges or diverges because that only depends on the value of *r* which we can determine regardless of the starting value of *n* with enough work. However, if the series does converge we won't be able to use the formula for determining the value of the series as that also needs the value of *a* and that does require the series to start at one of the two standard starting values.

We have two options for taking care of this problem. One is to use an index shift to convert this into a series that starts at one of the standard starting values of *n*. In most cases this is probably the only real option.

However, in this case let's notice that this series is almost identical to the series from the previous problem. The only difference is that this series starts at $n=2$ while the series in the previous problem starts at $n=1$. This means that we can use the results of the previous problem to greatly reduce the amount of work needed here.

Step 3
We know that the series in the previous problem converged and since we're only changing the starting value of *n* that will not affect the convergence of the series. Therefore, the series in this problem will also **converge**.

Since we also know that the value of the series in the previous series is $\frac{1225}{2}$ we can find the value of the series in this problem. All we need to do is strip out one term from the series in the previous problem to get,

$$\sum_{n=1}^{\infty}\frac{5^{n+1}}{7^{n-2}}=\frac{5^2}{7^{-1}}+\sum_{n=2}^{\infty}\frac{5^{n+1}}{7^{n-2}}$$

Then using the value we found in the previous problem can get the value of the series from this problem as follows,

$$\frac{1225}{2}=175+\sum_{n=2}^{\infty}\frac{5^{n+1}}{7^{n-2}} \qquad\Rightarrow\qquad \sum_{n=2}^{\infty}\frac{5^{n+1}}{7^{n-2}}=\frac{1225}{2}-175=\boxed{\frac{875}{2}}$$

On a quick side note if you did chose to do an index shift here are the two series (for each possible starting value of *n*) that you should have gotten.

$$\sum_{n=2}^{\infty}\frac{5^{n+1}}{7^{n-2}}=\sum_{n=1}^{\infty}\frac{5^{n+2}}{7^{n-1}}=\sum_{n=0}^{\infty}\frac{5^{n+3}}{7^{n}}$$

Both of the last two are in the "standard" form and can be used to arrive at the same result as above.

---

7. Determine if the series converges or diverges. If the series converges give its value.

$$\sum_{n=4}^{\infty}\frac{10}{n^2-4n+3}$$

Step 1

Given that all three of the special series we looked at in this section are all pretty distinct it is hopefully clear that this is not a geometric or harmonic series. That only leaves telescoping as a possibility.

Step 2
Now, we need to be careful here. There is no way to actually identify the series as a telescoping series at this point. We are only hoping that it is a telescoping series.

Therefore, the first real step here is to perform partial fractions on the series term to see what we get. Here is the partial fraction work for the series term.

$$\frac{10}{n^2-4n+3}=\frac{10}{(n-1)(n-3)}=\frac{A}{n-1}+\frac{B}{n-3} \qquad \rightarrow \qquad 10=A(n-3)+B(n-1)$$

$$\begin{aligned} n&=1 & 10&=-2A \\ n&=3 & 10&=2B \end{aligned} \qquad \rightarrow \qquad \begin{aligned} A&=-5 \\ B&=5 \end{aligned}$$

The series term in partial fraction form is then,

$$\frac{10}{n^2-4n+3}=\frac{5}{n-3}-\frac{5}{n-1}$$

Step 3
The partial sums for this series are then,

$$s_n=\sum_{i=4}^{n}\left[\frac{5}{i-3}-\frac{5}{i-1}\right]$$

Step 4
Expanding the partial sums from the previous step give,

$$\begin{aligned} s_n &= \sum_{i=4}^{n}\left[\frac{5}{i-3}-\frac{5}{i-1}\right] \\ &= \left[\frac{5}{1}-\frac{5}{3}\right]+\left[\frac{5}{2}-\frac{5}{4}\right]+\left[\frac{5}{3}-\frac{5}{5}\right]+\left[\frac{5}{4}-\frac{5}{6}\right]+\left[\frac{5}{5}-\frac{5}{7}\right]+\cdots \\ &\qquad +\left[\frac{5}{n-7}-\frac{5}{n-5}\right]+\left[\frac{5}{n-6}-\frac{5}{n-4}\right]+\left[\frac{5}{n-5}-\frac{5}{n-3}\right]+ \\ &\qquad \left[\frac{5}{n-4}-\frac{5}{n-2}\right]+\left[\frac{5}{n-3}-\frac{5}{n-1}\right] \\ &= 5+\frac{5}{2}-\frac{5}{n-2}-\frac{5}{n-1} \end{aligned}$$

It is important when doing this expanding to expand out from both the initial and final values of *i* and to expand out until all the parts of a series term cancel. Once that has been done it is safe to assume that the cancelling will continue until we get near the end of the expansion.

Also, as seen above these can be quite messy to expand out until everything starts to cancel out so don't get too excited about it when it does get messy like this. It just happens sometimes and we have to be careful with all the expansion.

Note that at this point we now know that the series was a telescoping series since we got almost all the "interior" terms to cancel out.

Step 5
At this point all we need to do is look at the limit of the partial sums to get,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left[\frac{15}{2} - \frac{5}{n-2} - \frac{5}{n-1}\right] = \frac{15}{2}$$

Step 6
The limit of the partial sums exists and is a finite number (*i.e.* not infinity) and so we can see that the series **converges** and its value is,

$$\sum_{n=4}^{\infty} \frac{10}{n^2 - 4n + 3} = \frac{15}{2}$$

Section 4-6 : Integral Test

1. Determine if the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$$

Solution
There really isn't all that much to this problem. We could use the Integral Test on this series or we could just use the *p*-series Test we discussed in the notes for this section.

We can clearly see that $p = \pi > 1$ and so by the *p*-series Test this series must **converge.**

---

2. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{2}{3+5n}$$

Step 1
Okay, prior to using the Integral Test on this series we first need to verify that we can in fact use the Integral Test!

Step 2
The series terms are,

$$a_n = \frac{2}{3+5n}$$

We can clearly see that for the range of *n* in the series the terms are positive and so that condition is met.

Step 3
In this case because there is only one *n* in the denominator and because all the terms in the denominator are positive it is (hopefully) clear that,

$$a_n = \frac{2}{3+5n} > \frac{2}{3+5\left(n+1\right)} = a_{n+1}$$

and so the series terms are decreasing.

Okay, we now know that both of the conditions required for us to use the Integral Test have been verified we can proceed with the Integral Test.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 4
Now, let's compute the integral for the test.

$$\int_0^\infty \frac{2}{3+5x}\,dx = \lim_{t\to\infty}\int_0^t \frac{2}{3+5x}\,dx = \lim_{t\to\infty}\left(\frac{2}{5}\ln\left|3+5x\right|\right)\Bigg|_0^t = \lim_{t\to\infty}\left(\frac{2}{5}\ln\left|3+5t\right| - \frac{2}{5}\ln\left|3\right|\right) = \infty$$

Step 5
Okay, the integral from the last step is a divergent integral and so by the Integral Test the series must also be a **divergent** series.

---

3. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty}\frac{1}{\left(2n+7\right)^3}$$

Step 1
Okay, prior to using the Integral Test on this series we first need to verify that we can in fact use the Integral Test!

Step 2
The series terms are,

$$a_n = \frac{1}{\left(2n+7\right)^3}$$

We can clearly see that for the range of *n* in the series the terms are positive and so that condition is met.

Step 3
In this case because there is only one *n* in the denominator and because all the terms in the denominator are positive it is (hopefully) clear that,

$$a_n = \frac{1}{\left(2n+7\right)^3} > \frac{1}{\left(2\left(n+1\right)+7\right)^3} = a_{n+1}$$

and so the series terms are decreasing.

Okay, we now know that both of the conditions required for us to use the Integral Test have been verified we can proceed with the Integral Test.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 4
Now, let's compute the integral for the test.

$$\int_2^\infty \frac{1}{(2x+7)^3}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{(2x+7)^3}\,dx = \lim_{t\to\infty}\left.\left(-\frac{1}{4}\frac{1}{(2x+7)^2}\right)\right|_2^t$$
$$= \lim_{t\to\infty}\left(-\frac{1}{4}\frac{1}{(2t+7)^2}+\frac{1}{4}\frac{1}{(11)^2}\right) = \frac{1}{484}$$

Step 5
Okay, the integral from the last step is a convergent integral and so by the Integral Test the series must also be a **convergent** series.

---

4. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty}\frac{n^2}{n^3+1}$$

Step 1
Okay, prior to using the Integral Test on this series we first need to verify that we can in fact use the Integral Test!

Step 2
The series terms are,

$$a_n = \frac{n^2}{n^3+1}$$

We can clearly see that for the range of n in the series the terms are positive and so that condition is met.

Step 3
In this case we need to be a little more careful with checking the decreasing condition. We can't just plug in $n + 1$ into the series term as we've done in the first couple of problems in this section. Doing that would suggest that both the numerator and denominator will increase and so it's not all that clear cut of a case that the terms will be decreasing.

Therefore, we'll need to do a quick Calculus I increasing/decreasing analysis. Here the function for the series terms and its derivative.

$$f(x) = \frac{x^2}{x^3+1} \qquad\qquad f'(x) = \frac{2x - x^4}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}$$

With a quick number line or sign chart we can see that the function will increase for $0 < x < \sqrt[3]{2} = 1.2599$ and will decrease for $\sqrt[3]{2} = 1.2599 < x < \infty$. Because the function and series terms are the same we know that the series terms will have the same increasing/decreasing behavior.

So, from this analysis we can see that the series terms are not always decreasing but will be decreasing for $n > \sqrt[3]{2}$ which is sufficient for us to use to say that this condition is also met.

Okay, we now know that both of the conditions required for us to use the Integral Test have been verified we can proceed with the Integral Test.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 4
Now, let's compute the integral for the test.

$$\int_0^\infty \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty}\int_0^t \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty}\left(\frac{1}{3}\ln\left|x^3+1\right|\right)\Bigg|_0^t = \lim_{t\to\infty}\left(\frac{1}{3}\ln\left|t^3+1\right| - \ln(1)\right) = \infty$$

Step 5
Okay, the integral from the last step is a divergent integral and so by the Integral Test the series must also be a **divergent** series.

---

5. Determine if the following series converges or diverges.

$$\sum_{n=3}^{\infty} \frac{3}{n^2 - 3n + 2}$$

Step 1
Okay, prior to using the Integral Test on this series we first need to verify that we can in fact use the Integral Test!

Step 2
The series terms are,

$$a_n = \frac{3}{n^2 - 3n + 2}$$

We can clearly see that for $n \ge 3$ (which matches our range of *n* for the series) we will have,

$$n^2 \ge 3n \qquad \Rightarrow \qquad n^2 - 3n \ge 0 \qquad \Rightarrow \qquad n^2 - 3n + 2 \ge n^2 - 3n \ge 0$$

Therefore, the series terms are positive and so that condition is met.

Note that on occasion we'll need to do more than just state that the series terms are positive by inspection and do a little work to show that the terms really are positive!

Step 3
In this case we need to be a little more careful with checking the decreasing condition. We can't just plug in *n* + 1 into the series term as we've done in the first couple of problems in this section.

Doing that the first term in the denominator would be getting larger which would suggest the series term is decreasing. However, because the second term in the denominator is subtracted off if we increase *n* that would suggest the denominator is getting larger and hence the series term is increasing.

Because we have these "competing" interests we'll need to do a quick Calculus I increasing/decreasing analysis. Here the function for the series terms and its derivative.

$$f\left(x\right) = \frac{3}{x^2 - 3x + 2} \qquad\qquad f'\left(x\right) = \frac{9 - 6x}{\left(x^2 - 3x + 2\right)^2}$$

With a quick number line or sign chart we can see that the function will increase for $x < \frac{3}{2}$ and will decrease for $x > \frac{3}{2}$. Because the function and series terms are the same we know that the series terms will have the same increasing/decreasing behavior.

So, from this analysis we can see that the series terms are always decreasing for the range *n* in our series and so this condition is also met.

Okay, we now know that both of the conditions required for us to use the Integral Test have been verified we can proceed with the Integral Test.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 4
Now, let's compute the integral for the test. The integral we'll need to compute is,

$$\int_3^\infty \frac{3}{x^2 - 3x + 2}\,dx$$

This integral will however require us to do some quick partial fractions in order to do the evaluation. Here is that quick work.

$$\frac{3}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \qquad \to \qquad 3 = A(x-2) + B(x-1)$$

$$\begin{array}{ll} x=1 & 3=-A \\ x=2 & 3=B \end{array} \qquad \Rightarrow \qquad \begin{array}{l} A=-3 \\ B=3 \end{array}$$

The integral is then,

$$\begin{aligned}\int_3^\infty \frac{3}{x-2} - \frac{3}{x-1}\,dx &= \lim_{t\to\infty}\int_3^t \frac{3}{x-2} - \frac{3}{x-1}\,dx = \lim_{t\to\infty}\left(3\ln|x-2| - 3\ln|x-1|\right)\Big|_3^t \\ &= \lim_{t\to\infty}\left[3\ln|t-2| - 3\ln|t-1| - \left(3\ln|1| - 3\ln|2|\right)\right] \\ &= \lim_{t\to\infty}\left[3\ln\left|\frac{t-2}{t-1}\right| + 3\ln|2|\right] = 3\ln\left(\frac{1}{1}\right) + 3\ln(2) = 3\ln(2)\end{aligned}$$

Be careful with the limit of the first two terms! To correctly compute the limit they need to be combined using logarithm properties as shown and we can then do a L'Hospital's Rule on the argument of the log to compute the limit.

Step 5

Okay, the integral from the last step is a convergent integral and so by the Integral Test the series must also be a **convergent** series.

Section 4-7 : Comparison Test/Limit Comparison Test

1. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+1\right)^2$$

Step 1
First, the series terms are,

$$a_n=\left(\frac{1}{n^2}+1\right)^2$$

and it should pretty obvious in this case that they are positive and so we know that we can use the Comparison Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2
For most of the Comparison Test problems we usually guess the convergence and proceed from there. However, in this case it is hopefully clear that for any *n*,

$$\left(\frac{1}{n^2}+1\right)^2>\left(1\right)^2=1$$

Now, let's take a look at the following series,

$$\sum_{n=1}^{\infty}1$$

Because $\lim\limits_{n\to\infty}1=1\neq 0$ we can see from the Divergence Test that this series will be divergent.

So we've found a divergent series with terms that are smaller than the original series terms. Therefore, by the Comparison Test the series in the problem statement must also be **divergent**.

As a final note for this problem notice that we didn't actually need to do a Comparison Test to arrive at this answer. We could have just used the Divergence Test from the beginning since,

$$\lim_{n\to\infty}\left(\frac{1}{n^2}+1\right)^2 = 1 \neq 0$$

This is something that you should always keep in mind with series convergence problems. The Divergence Test is a quick test that can, on occasion, be used to quickly determine that a series diverges and hence avoid a lot of the hassles of some of the other tests.

---

2. Determine if the following series converges or diverges.

$$\sum_{n=4}^{\infty}\frac{n^2}{n^3-3}$$

Step 1
First, the series terms are,

$$a_n = \frac{n^2}{n^3-3}$$

and it should pretty obvious that as long as $n > \sqrt{3}$ (which we'll always have for this series) that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The "-3" in the denominator won't really affect the size of the denominator for large enough *n* and so it seems like for large *n* that the term will probably behave like,

$$b_n = \frac{n^2}{n^3} = \frac{1}{n}$$

We also know that the series,

$$\sum_{n=4}^{\infty}\frac{1}{n}$$

will diverge because it is a harmonic series or by the *p*-series Test.

Therefore, it makes some sense that we can guess the series in the problem statement will probably diverge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series diverges we'll need to find a series with smaller terms that we know, or can prove, diverges.

Note as well that we'll also need to prove that the new series terms really are smaller than the terms from the series in the problem statement. We can't just "hope" that the will be smaller.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms smaller by either making the numerator smaller or the denominator larger.

In this case it should be pretty clear that,

$$n^3 > n^3 - 3$$

Therefore, we'll have the following relationship.

$$\frac{n^2}{n^3} < \frac{n^2}{n^3 - 3}$$

You do agree with this right? The numerator in each is the same while the denominator in the left term is larger than the denominator in the right term. Therefore, the rational expression on the left must be smaller than the rational expression on the right.

Step 4
Now, the series,

$$\sum_{n=4}^{\infty} \frac{n^2}{n^3} = \sum_{n=4}^{\infty} \frac{1}{n}$$

is a divergent series (as discussed above) and we've also shown that the series terms in this series are smaller than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **diverge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

3. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{7}{n(n+1)}$$

Step 1
First, the series terms are,

$$a_n = \frac{7}{n(n+1)}$$

and it should pretty obvious that for the range of *n* we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The "+1" in the denominator won't really affect the size of the denominator for large enough *n* and so it seems like for large *n* that the term will probably behave like,

$$b_n = \frac{7}{n(n)} = \frac{7}{n^2}$$

We also know that the series,

$$\sum_{n=2}^{\infty} \frac{7}{n^2}$$

will converge by the *p*-series Test ($p = 2 > 1$).

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3

So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

In this case it should be pretty clear that,

$$n < n+1 \qquad \Rightarrow \qquad n(n) < n(n+1)$$

Therefore, we'll have the following relationship.

$$\frac{7}{n(n)} > \frac{7}{n(n+1)}$$

You do agree with this right? The numerator in each is the same while the denominator in the left term is smaller than the denominator in the right term. Therefore, the rational expression on the left must be larger than the rational expression on the right.

Step 4
Now, the series,

$$\sum_{n=2}^{\infty} \frac{7}{n(n)} = \sum_{n=2}^{\infty} \frac{7}{n^2}$$

is a convergent series (as discussed above) and we've also shown that the series terms in this series are larger than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **converge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

---

4. Determine if the following series converges or diverges.

$$\sum_{n=7}^{\infty} \frac{4}{n^2 - 2n - 3}$$

Step 1
First, the series terms are,

$$a_n = \frac{4}{n^2 - 2n - 3}$$

You can verify that for $n \ge 7$ we have $n^2 > 2n + 3$ and so $n^2 - 2n - 3 = n^2 - \left(2n + 3\right) > 0$. Therefore, the series terms are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

For large enough *n* we know that the n^2 (a quadratic term) in the denominator will increase at a much faster rate than the $-2n - 3$ (a linear term) portion of the denominator. Therefore the n^2 portion of the denominator will, in all likelihood, define the behavior of the denominator and so the terms should behave like,

$$b_n = \frac{4}{n^2}$$

We also know that the series,

$$\sum_{n=4}^{\infty} \frac{4}{n^2}$$

will converge by the *p*-series Test ($p = 2 > 1$).

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

We now have a problem however. The obvious thing to try is to drop the last two terms on the denominator. Doing that however gives the following inequality,

$$n^2 > n^2 - 2n - 3$$

This in turn gives the following relationship.

$$\frac{4}{n^2} < \frac{4}{n^2 - 2n - 3}$$

The denominator on the left is larger and so the rational expression on the left must be smaller. This leads to the problem. While the series,

$$\sum_{n=4}^{\infty} \frac{4}{n^2}$$

will definitely converge (as discussed above) it's terms are smaller than the series terms in the problem statement. Just because a series with smaller terms converges does not, in any way, imply a series with larger terms will also converge!

There are other manipulations we might try but they are all liable to run into similar issues or end up with new terms that we wouldn't be able to quickly prove convergence on.

Hint : So, if the Comparison Test won't easily work what else is there to do?

Step 4
So, the Comparison Test won't easily work in this case. That pretty much leaves the Limit Comparison Test to try. This test only requires positive terms (which we have) and a second series that we're pretty sure behaves like the series we want to know the convergence for. Note as well that, for the Limit Comparison Test, we don't care if the terms for the second series are larger or smaller than problem statement series terms.

If you think about it we already have exactly what we need. In Step 2 we used a second series to guess at the convergence of the problem statement series. The terms in the new series are positive (which we need) and we're pretty sure it behaves in the same manner as the problem statement series.

So, let's compute the limit we need for the Limit Comparison Test.

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left[a_n \frac{1}{b_n} \right] = \lim_{n\to\infty} \left[\frac{4}{n^2 - 2n - 3} \frac{n^2}{4} \right] = \lim_{n\to\infty} \left[\frac{n^2}{n^2 - 2n - 3} \right] = 1$$

Step 5
Okay. We now have $0 < c = 1 < \infty$, *i.e.* c is not zero or infinity and so by the Limit Comparison Test the two series must have the same convergence. We determined in Step 2 that the second series converges and so the series given in the problem statement must also **converge**.

Be careful with the Comparison Test. Too often students just try to "force" larger or smaller by just hoping that the second series terms has the correct relationship (*i.e.* larger or smaller as needed) to the problem series terms. The problem is that this often leads to an incorrect answer. Be careful to always prove the larger/smaller nature of the series terms and if you can't get a series term of the correct larger/smaller nature then you may need to resort to the Limit Comparison Test.

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5. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{n-1}{\sqrt{n^6+1}}$$

Step 1
First, the series terms are,

$$a_n = \frac{n-1}{\sqrt{n^6+1}}$$

and it should pretty obvious that for the range of n we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The "-1" in the numerator and the "+1" in the denominator won't really affect the size of the numerator and denominator respectively for large enough n and so it seems like for large n that the term will probably behave like,

$$b_n = \frac{n}{\sqrt{n^6}} = \frac{n}{n^3} = \frac{1}{n^2}$$

We also know that the series,

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

will converge by the p-series Test ($p = 2 > 1$).

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

In this case we can work with both the numerator and the denominator. Let's start with the numerator. It should be pretty clear that,

$$n > n-1$$

Using this we can make the numerator larger to get the following relationship,

$$\frac{n-1}{\sqrt{n^6+1}} < \frac{n}{\sqrt{n^6+1}}$$

Now, in the denominator it again is hopefully clear that,

$$n^6 < n^6 + 1$$

Using this we can make the denominator smaller (and hence make the rational expression larger) to get,

$$\frac{n-1}{\sqrt{n^6+1}} < \frac{n}{\sqrt{n^6+1}} < \frac{n}{\sqrt{n^6}} = \frac{1}{n^2}$$

Step 4
Now, the series,

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

is a convergent series (as discussed above) and we've also shown that the series terms in this series are larger than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **converge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

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6. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2n^3 + 7}{n^4 \sin^2\left(n\right)}$$

Step 1
First, the series terms are,

$$a_n = \frac{2n^3 + 7}{n^4 \sin^2\left(n\right)}$$

and it should pretty obvious that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The "+7" in the numerator and the "$\sin^2\left(n\right)$" in the denominator won't really affect the size of the numerator and denominator respectively for large enough *n* and so it seems like for large *n* that the term will probably behave like,

$$b_n = \frac{2n^3}{n^4} = \frac{2}{n}$$

We also know that the series,

$$\sum_{n=1}^{\infty} \frac{2}{n}$$

will diverge because it is a harmonic series or by the *p*-series Test.

Therefore, it makes some sense that we can guess the series in the problem statement will probably diverge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series diverges we'll need to find a series with smaller terms that we know, or can prove, diverges.

Note as well that we'll also need to prove that the new series terms really are smaller than the terms from the series in the problem statement. We can't just "hope" that the will be smaller.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms smaller by either making the numerator smaller or the denominator larger.

In this case we can work with both the numerator and the denominator. Let's start with the numerator. It should be pretty clear that,

$$2n^3 < 2n^3 + 7$$

Using this we can make the numerator smaller to get the following relationship,

$$\frac{2n^3+7}{n^4\sin^2\left(n\right)} > \frac{2n^3}{n^4\sin^2\left(n\right)}$$

Now, we know that $0 \le \sin^2\left(n\right) \le 1$ and so in the denominator we can see that if we replace the $\sin^2\left(n\right)$ with its largest possible value we have,

$$n^4\sin^2\left(n\right) < n^4\left(1\right) = n^4$$

Using this we can make the denominator larger (and hence make the rational expression smaller) to get,

$$\frac{2n^3+7}{n^4\sin^2\left(n\right)} > \frac{2n^3}{n^4\sin^2\left(n\right)} > \frac{2n^3}{n^4} = \frac{2}{n}$$

Step 4
Now, the series,

$$\sum_{n=1}^{\infty}\frac{2}{n}$$

is a divergent series (as discussed above) and we've also shown that the series terms in this series are smaller than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **diverge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

---

7. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty}\frac{2^n\sin^2(5n)}{4^n+\cos^2(n)}$$

Step 1
First, the series terms are,

$$a_n=\frac{2^n\sin^2(5n)}{4^n+\cos^2(n)}$$

and it should pretty obvious that for the range of *n* we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The trig functions in the numerator and in the denominator won't really affect the size of the numerator and denominator for large enough *n* and so it seems like for large *n* that the term will probably behave like,

$$b_n=\frac{2^n}{4^n}=\left(\frac{2}{4}\right)^n=\left(\frac{1}{2}\right)^n$$

We also know that the series,

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$$

will converge because it is a geometric series with $r=\frac{1}{2}<1$.

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

In this case we can work with both the numerator and the denominator. Let's start with the numerator. We know that $0 \le \sin^2(5n) \le 1$ and so replacing the $\sin^2(5n)$ in the numerator with the largest possible value we get,

$$2^n \sin^2(5n) < 2^n(1) = 2^n$$

Using this we can make the numerator larger to get the following relationship,

$$\frac{2^n \sin^2(5n)}{4^n + \cos^2(n)} < \frac{2^n}{4^n + \cos^2(n)}$$

Now, in the denominator we know that $0 \le \cos^2(n) \le 1$ and so replacing the $\cos^2(n)$ with the smallest possible value we get,

$$4^n + \cos^2(n) > 4^n + 0 = 4^n$$

Using this we can make the denominator smaller (and hence make the rational expression larger) to get,

$$\frac{2^n \sin^2(5n)}{4^n + \cos^2(n)} < \frac{2^n}{4^n + \cos^2(n)} < \frac{2^n}{4^n} = \left(\frac{1}{2}\right)^n$$

Step 4
Now, the series,

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

is a convergent series (as discussed above) and we've also shown that the series terms in this series are larger than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **converge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

---

8. Determine if the following series converges or diverges.

$$\sum_{n=3}^{\infty} \frac{\mathbf{e}^{-n}}{n^2 + 2n}$$

Step 1
First, the series terms are,

$$a_n = \frac{\mathbf{e}^{-n}}{n^2 + 2n}$$

and it should pretty obvious that for the range of *n* we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
In this case let's first notice the exponential in the numerator will go to zero as *n* goes to infinity. Let's also notice that the denominator is just a polynomial. In cases like this the exponential is going to go to zero so fast that behavior of the denominator will not matter at all and in all probability the series in the problem statement will probably converge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

In this case we can work with both the numerator and the denominator. Let's start with the numerator. We can use some quick Calculus I to prove that $\mathbf{e}^{-n}$ is a decreasing function and so,

$$\mathbf{e}^{-n} < \mathbf{e}^{-3} < 1$$

Using this we can make the numerator larger to get the following relationship,

$$\frac{\mathbf{e}^{-n}}{n^2 + 2n} < \frac{1}{n^2 + 2n}$$

Now, in the denominator it should be fairly clear that,

$$n^2 + 2n > n^2$$

Using this we can make the denominator smaller (and hence make the rational expression larger) to get,

$$\frac{\mathbf{e}^{-n}}{n^2 + 2n} < \frac{1}{n^2 + 2n} < \frac{1}{n^2}$$

Step 4
Now, the series,

$$\sum_{n=3}^{\infty} \frac{1}{n^2}$$

is a convergent series (*p*-series Test with $p = 2 > 1$) and we've also shown that the series terms in this series are larger than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **converge**.

---

9. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{4n^2 - n}{n^3 + 9}$$

Step 1
First, the series terms are,

$$a_n = \frac{4n^2 - n}{n^3 + 9}$$

You can verify that for $n \ge 1$ we have $4n^2 > n$ and so $4n^2 - n > 0$. Therefore, the series terms are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

For large enough *n* we know that the n^2 (a quadratic term) in the numerator will increase at a much faster rate than the $-n$ (a linear term) portion of the numerator. Therefore the n^2 portion of the numerator will, in all likelihood, define the behavior of the numerator. Likewise, the "+9" in the denominator will not affect the size of the denominator for large *n* and so the terms should behave like,

$$b_n = \frac{4n^2}{n^3} = \frac{4}{n}$$

We also know that the series,

$$\sum_{n=1}^{\infty} \frac{4}{n}$$

will diverge because it is a harmonic series or by the *p*-series Test.

Therefore, it makes some sense that we can guess the series in the problem statement will probably diverge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series diverge we'll need to find a series with smaller terms that we know, or can prove, diverge.

Note as well that we'll also need to prove that the new series terms really are smaller than the terms from the series in the problem statement. We can't just "hope" that the will be smaller.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms smaller by either making the numerator smaller or the denominator larger.

We now have a problem however. The obvious thing to try is to drop the last term in both the numerator and the denominator. Doing that however gives the following inequalities,

$$4n^2 - n < 4n^2 \qquad\qquad n^3 + 9 > n^3$$

Using these two in the series terms gives the following relationship,

$$\frac{4n^2 - n}{n^3 + 9} < \frac{4n^2}{n^3 + 9} < \frac{4n^2}{n^3} = \frac{4}{n}$$

Now the series,

$$\sum_{n=0}^{\infty} \frac{4}{n}$$

will definitely diverge (as discussed above) it's terms are larger than the series terms in the problem statement. Just because a series with larger terms diverges does not, in any way, imply a series with smaller terms will also diverge!

There are other manipulations we might try but they are all liable to run into similar issues or end up with new terms that we wouldn't be able to quickly prove convergence on.

Hint : So, if the Comparison Test won't easily work what else is there to do?

Step 4
So, the Comparison Test won't easily work in this case. That pretty much leaves the Limit Comparison Test to try. This test only requires positive terms (which we have) and a second series that we're pretty sure behaves like the series we want to know the convergence for. Note as well that, for the Limit Comparison Test, we don't care if the terms for the second series are larger or smaller than problem statement series terms.

If you think about it we already have exactly what we need. In Step 2 we used a second series to guess at the convergence of the problem statement series. The terms in the new series are positive (which we need) and we're pretty sure it behaves in the same manner as the problem statement series.

So, let's compute the limit we need for the Limit Comparison Test.

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left[a_n \frac{1}{b_n}\right] = \lim_{n\to\infty}\left[\frac{4n^2 - n}{n^3 + 9}\,\frac{n}{4}\right] = \lim_{n\to\infty}\left[\frac{4n^3 - n^2}{4n^3 + 36}\right] = 1$$

Step 5
Okay. We now have $0 < c = 1 < \infty$, *i.e.* c is not zero or infinity and so by the Limit Comparison Test the two series must have the same convergence. We determined in Step 2 that the second series diverges and so the series given in the problem statement must also **diverge**.

Be careful with the Comparison Test. Too often students just try to "force" larger or smaller by just hoping that the second series terms has the correct relationship (*i.e.* larger or smaller as needed) to the problem series terms. The problem is that this often leads to an incorrect answer. Be careful to always prove the larger/smaller nature of the series terms and if you can't get a series term of the correct larger/smaller nature then you may need to resort to the Limit Comparison Test.

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10. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\sqrt{2n^2 + 4n + 1}}{n^3 + 9}$$

Step 1
First, the series terms are,

$$a_n = \frac{\sqrt{2n^2 + 4n + 1}}{n^3 + 9}$$

and it should pretty obvious that for the range of *n* we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

For large enough *n* we know that the $2n^2$ (a quadratic term) in the numerator will increase at a much faster rate than the $4n+1$ (a linear term) portion of the numerator. Therefore the $2n^2$ portion of the numerator will, in all likelihood, define the behavior of the numerator. Likewise, the "+9" in the denominator will not affect the size of the denominator for large *n* and so the terms should behave like,

$$b_n = \frac{\sqrt{2n^2}}{n^3} = \frac{\sqrt{2}}{n^2}$$

We also know that the series,

$$\sum_{n=1}^{\infty} \frac{\sqrt{2}}{n^2}$$

will converge by the *p*-series Test ($p = 2 > 1$).

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

We now have a problem however. The obvious thing to try is to drop the last two terms in the numerator and the last term in the denominator. Doing that however gives the following inequalities,

$$2n^2 < 2n^2 + 4n + 1 \qquad\qquad n^3 + 9 > n^3$$

This leads to a real problem! If we use the inequality for the numerator we're going to get a smaller rational expression and if we use the inequality for the denominator we're going to get a larger rational expression. Because these two can't both be used at the same time it will make it fairly difficult to use the Comparison Test since neither one individually give a series we can quickly deal with.

Hint : So, if the Comparison Test won't easily work what else is there to do?

Step 4
So, the Comparison Test won't easily work in this case. That pretty much leaves the Limit Comparison Test to try. This test only requires positive terms (which we have) and a second series that we're pretty sure behaves like the series we want to know the convergence for. Note as well that, for the Limit Comparison Test, we don't care if the terms for the second series are larger or smaller than problem statement series terms.

If you think about it we already have exactly what we need. In Step 2 we used a second series to guess at the convergence of the problem statement series. The terms in the new series are positive (which we need) and we're pretty sure it behaves in the same manner as the problem statement series.

So, let's compute the limit we need for the Limit Comparison Test.

$$\begin{aligned} c = \lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\left[a_n\,\frac{1}{b_n}\right] = \lim_{n\to\infty}\left[\frac{\sqrt{2n^2+4n+1}}{n^3+9}\,\frac{n^2}{\sqrt{2}}\right] \\ &= \lim_{n\to\infty}\left[\frac{n^2\sqrt{n^2\left(2+\frac{4}{n}+\frac{1}{n^2}\right)}}{\sqrt{2}\,n^3\left(1+\frac{9}{n^3}\right)}\right] = \lim_{n\to\infty}\left[\frac{n^2\left(n\right)\sqrt{2+\frac{4}{n}+\frac{1}{n^2}}}{\sqrt{2}\,n^3\left(1+\frac{9}{n^3}\right)}\right] = \frac{\sqrt{2}}{\sqrt{2}} = 1 \end{aligned}$$

Step 5
Okay. We now have $0 < c = 1 < \infty$, *i.e.* c is not zero or infinity and so by the Limit Comparison Test the two series must have the same convergence. We determined in Step 2 that the second series converges and so the series given in the problem statement must also **converge**.

Be careful with the Comparison Test. Too often students just try to "force" larger or smaller by just hoping that the second series terms has the correct relationship (*i.e.* larger or smaller as needed) to the problem series terms. The problem is that this often leads to an incorrect answer. Be careful to always prove the larger/smaller nature of the series terms and if you can't get a series term of the correct larger/smaller nature then you may need to resort to the Limit Comparison Test.

Section 4-8 : Alternating Series Test

1. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{7+2n}$$

Step 1
First, this is (hopefully) clearly an alternating series with,

$$b_n = \frac{1}{7+2n}$$

and it should pretty obvious the b_n are positive and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2
Let's first take a look at the limit,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{7+2n} = 0$$

So, the limit is zero and so the first condition is met.

Step 3
Now let's take care of the decreasing check. In this case it should be pretty clear that,

$$\frac{1}{7+2n} > \frac{1}{7+2(n+1)}$$

since increasing *n* will only increase the denominator and hence force the rational expression to be smaller.

Therefore the b_n form a decreasing sequence.

Step 4
So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

---

2. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+3}}{n^3 + 4n + 1}$$

Step 1
First, this is (hopefully) clearly an alternating series with,

$$b_n = \frac{1}{n^3 + 4n + 1}$$

and it should pretty obvious the b_n are positive and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2
Let's first take a look at the limit,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^3 + 4n + 1} = 0$$

So, the limit is zero and so the first condition is met.

Step 3
Now let's take care of the decreasing check. In this case it should be pretty clear that,

$$\frac{1}{n^3 + 4n + 1} > \frac{1}{(n+1)^3 + 4(n+1) + 1}$$

since increasing *n* will only increase the denominator and hence force the rational expression to be smaller.

Therefore the b_n form a decreasing sequence.

Step 4
So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

---

3. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{1}{(-1)^n \left(2^n + 3^n\right)}$$

Step 1

Do not get excited about the $(-1)^n$ is in the denominator! This is still an alternating series! All the $(-1)^n$ does is change the sign regardless of whether or not it is in the numerator.

Also note that we could just as easily rewrite the terms as,

$$\frac{1}{(-1)^n\left(2^n+3^n\right)}=\frac{(-1)^n}{(-1)^n}\frac{1}{(-1)^n\left(2^n+3^n\right)}=\frac{(-1)^n}{(-1)^{2n}\left(2^n+3^n\right)}=\frac{(-1)^n}{\left(2^n+3^n\right)}$$

Note that $(-1)^{2n}=1$ because the exponent is always even!

So, we now know that this is an alternating series with,

$$b_n=\frac{1}{2^n+3^n}$$

and it should pretty obvious the b_n are positive and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2

Let's first take a look at the limit,

$$\lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{1}{2^n+3^n}=0$$

So, the limit is zero and so the first condition is met.

Step 3

Now let's take care of the decreasing check. In this case it should be pretty clear that,

$$\frac{1}{2^n+3^n}>\frac{1}{2^{n+1}+3^{n+1}}$$

since increasing *n* will only increase the denominator and hence force the rational expression to be smaller.

Therefore the b_n form a decreasing sequence.

Step 4

So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

---

4. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+6} n}{n^2+9}$$

Step 1
First, this is (hopefully) clearly an alternating series with,

$$b_n = \frac{n}{n^2+9}$$

and it should pretty obvious the b_n are positive and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2
Let's first take a look at the limit,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+9} = 0$$

So, the limit is zero and so the first condition is met.

Step 3
Now let's take care of the decreasing check. In this case increasing *n* will increase both the numerator and denominator and so we can't just say that clearly the terms are decreasing as we did in the first few problems.

We will have no choice but to do a little Calculus I work for this problem. Here is the function and derivative for that work.

$$f(x) = \frac{x}{x^2+9} \qquad f'(x) = \frac{9-x^2}{\left(x^2+9\right)^2}$$

It should be pretty clear that the function will be increasing in $0 \le x < 3$ and decreasing in $x > 3$ (the range of *x* that corresponds to our range of *n*).

So, the b_n do not actually form a decreasing sequence but they are decreasing for $n > 3$ and so we can say that they are eventually decreasing and as discussed in the notes that will be sufficient for us.

Step 4
So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

---

5. Determine if the following series converges or diverges.

$$\sum_{n=4}^{\infty} \frac{(-1)^{n+2}(1-n)}{3n-n^2}$$

Step 1
First, this is (hopefully) clearly an alternating series with,

$$b_n = \frac{1-n}{3n-n^2}$$

and b_n are positive for $n \geq 4$ and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2
Let's first take a look at the limit,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1-n}{3n-n^2} = 0$$

So, the limit is zero and so the first condition is met.

Step 3
Now let's take care of the decreasing check. In this case increasing *n* will increase both the numerator and denominator and so we can't just say that clearly the terms are decreasing as we did in the first few problems.

We will have no choice but to do a little Calculus I work for this problem. Here is the function and derivative for that work.

$$f(x) = \frac{1-x}{3x-x^2} \qquad\qquad f'(x) = \frac{-x^2+2x-3}{\left(3x-x^2\right)^2}$$

The numerator of the derivative is never zero for any real number (we'll leave that to you to verify) and since it is clearly negative at $x = 0$ we know that the function will always be decreasing for $x \geq 4$.

Therefore the b_n form a decreasing sequence.

Step 4
So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

Section 4-9 : Absolute Convergence

1. Determine if the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^3+1}$$

Step 1

Okay, let's first see if the series converges or diverges if we put absolute value on the series terms.

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^{n+1}}{n^3+1} \right| = \sum_{n=2}^{\infty} \frac{1}{n^3+1}$$

Now, notice that,

$$\frac{1}{n^3+1} < \frac{1}{n^3}$$

and we know by the *p*-series test that

$$\sum_{n=2}^{\infty} \frac{1}{n^3}$$

converges.

Therefore, by the Comparison Test we know that the series from the problem statement,

$$\sum_{n=2}^{\infty} \frac{1}{n^3+1}$$

will also converge.

Step 2

So, because the series with the absolute value converges we know that the series in the problem statement is **absolutely convergent**.

---

2. Determine if the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-3}}{\sqrt{n}}$$

Step 1

Okay, let's first see if the series converges or diverges if we put absolute value on the series terms.

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-3}}{\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}$$

Now, by the by the *p*-series test we can see that this series will diverge.

Step 2
So, at this point we know that the series in the problem statement is not absolutely convergent so all we need to do is check to see if it’s conditionally convergent or divergent. To do this all we need to do is check the convergence of the series in the problem statement.

The series in the problem statement is an alternating series with,

$$b_n = \frac{1}{\sqrt{n}}$$

Clearly the b_n are positive so we can use the Alternating Series Test on this series. It is hopefully clear that the b_n are a decreasing sequence and $\lim_{n\to\infty} b_n = 0$.

Therefore, by the Alternating Series Test the series from the problem statement is convergent.

Step 3
So, because the series with the absolute value diverges and the series from the problem statement converges we know that the series in the problem statement is **conditionally convergent**.

---

3. Determine if the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=3}^{\infty}\frac{(-1)^{n+1}(n+1)}{n^3+1}$$

Step 1
Okay, let’s first see if the series converges or diverges if we put absolute value on the series terms.

$$\sum_{n=3}^{\infty}\left|\frac{(-1)^{n+1}(n+1)}{n^3+1}\right| = \sum_{n=3}^{\infty}\frac{n+1}{n^3+1}$$

We know by the *p*-series test that the following series converges.

$$\sum_{n=3}^{\infty}\frac{1}{n^2}$$

If we now compute the following limit,

$$c = \lim_{n \to \infty} \left[\frac{n+1}{n^3+1} \frac{n^2}{1} \right] = \lim_{n \to \infty} \left[\frac{n^3+n^2}{n^3+1} \right] = 1$$

we know by the Limit Comparison Test that the two series in this problem have the same convergence because c is neither zero or infinity and because $\sum_{n=3}^{\infty} \frac{1}{n^2}$ converges we know that the series from the problem statement must also converge.

Step 2
So, because the series with the absolute value converges we know that the series in the problem statement is **absolutely convergent**.

Section 4-10 : Ratio Test

1. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3^{1-2n}}{n^2+1}$$

Step 1
We'll need to compute *L*.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{1-2(n+1)}}{(n+1)^2+1}\frac{n^2+1}{3^{1-2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3^{-1-2n}}{(n+1)^2+1}\frac{n^2+1}{3^{1-2n}}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)^2+1}\frac{n^2+1}{3^2}\right| = \lim_{n\to\infty}\left|\frac{n^2+1}{9\left[(n+1)^2+1\right]}\right| = \frac{1}{9}$$

When computing a_{n+1} be careful to pay attention to any coefficients of *n* and powers of *n*. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2
Okay, we can see that $L = \frac{1}{9} < 1$ and so by the Ratio Test the series **converges**.

---

2. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{(2n)!}{5n+1}$$

Step 1
We'll need to compute *L*.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2(n+1))!}{5(n+1)+1}\frac{5n+1}{(2n)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2n+2)!}{5n+6}\frac{5n+1}{(2n)!}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)(2n)!}{5n+6}\frac{5n+1}{(2n)!}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)(5n+1)}{5n+6}\right| = \infty$$

When computing a_{n+1} be careful to pay attention to any coefficients of *n* and powers of *n*. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2
Okay, we can see that $L = \infty > 1$ and so by the Ratio Test the series **diverges**.

---

3. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{(-2)^{1+3n}(n+1)}{n^2 5^{1+n}}$$

Step 1
We'll need to compute *L*.

$$\begin{aligned} L &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{1+3(n+1)}(n+1+1)}{(n+1)^2 5^{1+n+1}}\frac{n^2 5^{1+n}}{(-2)^{1+3n}(n+1)}\right| \\ &= \lim_{n\to\infty}\left|\frac{(-2)^{4+3n}(n+2)}{(n+1)^2 5^{2+n}}\frac{n^2 5^{1+n}}{(-2)^{1+3n}(n+1)}\right| = \lim_{n\to\infty}\left|\frac{(-2)^3(n+2)}{(n+1)^2(5)}\frac{n^2}{(n+1)}\right| \\ &= \lim_{n\to\infty}\left|\frac{-8n^2(n+2)}{5(n+1)^2(n+1)}\right| = \frac{8}{5} \end{aligned}$$

When computing a_{n+1} be careful to pay attention to any coefficients of *n* and powers of *n*. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2
Okay, we can see that $L = \frac{8}{5} > 1$ and so by the Ratio Test the series **diverges**.

---

4. Determine if the following series converges or diverges.

$$\sum_{n=3}^{\infty} \frac{\mathbf{e}^{4n}}{(n-2)!}$$

Step 1
We'll need to compute *L*.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\mathbf{e}^{4(n+1)}}{(n+1-2)!}\frac{(n-2)!}{\mathbf{e}^{4n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\mathbf{e}^{4n+4}}{(n-1)!}\frac{(n-2)!}{\mathbf{e}^{4n}}\right| = \lim_{n\to\infty}\left|\frac{\mathbf{e}^{4n+4}}{(n-1)(n-2)!}\frac{(n-2)!}{\mathbf{e}^{4n}}\right| = \lim_{n\to\infty}\left|\frac{\mathbf{e}^{4}}{n-1}\right| = 0$$

When computing a_{n+1} be careful to pay attention to any coefficients of *n* and powers of *n*. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2
Okay, we can see that $L = 0 < 1$ and so by the Ratio Test the series **converges**.

---

5. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{6n+7}$$

Step 1
We'll need to compute *L*.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1+1}}{6(n+1)+7}\frac{6n+7}{(-1)^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)^{n+2}}{6n+13}\frac{6n+7}{(-1)^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)(6n+7)}{6n+13}\right| = 1$$

When computing a_{n+1} be careful to pay attention to any coefficients of *n* and powers of *n*. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2
Okay, we can see that $L = 1$ and so by the Ratio Test tells us nothing about this series.

Step 3
Just because the Ratio Test doesn't tell us anything doesn't mean we can't determine if this series converges or diverges.

In fact, it's actually quite simple to do in this case. This is an Alternating Series with,

$$b_n = \frac{1}{6n+7}$$

The b_n are clearly positive and it should be pretty obvious (hopefully) that they also form a decreasing sequence. Finally, we also can see that $\lim\limits_{n\to\infty} b_n = 0$ and so by the Alternating Series Test this series will **converge**.

Note, that if this series were not in this section doing this as an Alternating Series from the start would probably have been the best way of approaching this problem.

Section 4-11 : Root Test

1. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty}\left(\frac{3n+1}{4-2n}\right)^{2n}$$

Step 1
We'll need to compute *L*.

$$L=\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\left|\left(\frac{3n+1}{4-2n}\right)^{2n}\right|^{\frac{1}{n}}=\lim_{n\to\infty}\left|\left(\frac{3n+1}{4-2n}\right)^{2}\right|=\left(-\frac{3}{2}\right)^{2}=\frac{9}{4}$$

Step 2
Okay, we can see that $L=\frac{9}{4}>1$ and so by the Root Test the series **diverges**.

---

2. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty}\frac{n^{1-3n}}{4^{2n}}$$

Step 1
We'll need to compute *L*.

$$L=\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\left|\frac{n^{1-3n}}{4^{2n}}\right|^{\frac{1}{n}}=\lim_{n\to\infty}\left|\frac{n^{\frac{1}{n}-3}}{4^{2}}\right|=\left|\frac{n^{\frac{1}{n}}\ n^{-3}}{4^{2}}\right|=\frac{(1)(0)}{16}=0$$

Step 2
Okay, we can see that $L=0<1$ and so by the Root Test the series **converges**.

---

3. Determine if the following series converges or diverges.

$$\sum_{n=4}^{\infty}\frac{(-5)^{1+2n}}{2^{5n-3}}$$

Step 1
We'll need to compute *L*.

$$L = \lim_{n\to\infty} \sqrt[n]{\left|a_n\right|} = \lim_{n\to\infty}\left|\frac{\left(-5\right)^{1+2n}}{2^{5n-3}}\right|^{\frac{1}{n}} = \lim_{n\to\infty}\left|\frac{\left(-5\right)^{\frac{1}{n}+2}}{2^{5-\frac{3}{n}}}\right| = \left|\frac{\left(-5\right)^2}{2^5}\right| = \frac{25}{32}$$

Step 2
Okay, we can see that $L = \frac{25}{32} < 1$ and so by the Root Test the series **converges**.

Section 4-12 : Strategy for Series

Problems have not yet been written for this section.

I was finding it very difficult to come up with a good mix of "new" problems and decided my time was better spent writing problems for later sections rather than trying to come up with a sufficient number of problems for what is essentially a review section. I intend to come back at a later date when I have more time to devote to this section and add problems then.

Section 4-13 : Estimating the Value of a Series

1. Use the Integral Test and $n = 10$ to estimate the value of $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\left(n^2+1\right)^2}$.

Step 1
Since we are being asked to use the Integral Test to estimate the value of the series we should first make sure that the Integral Test can actually be used on this series.

First, the series terms are clearly positive so that condition is met.

Now, let's do a little Calculus I on the following function.

$$f\left(x\right) = \frac{x}{\left(x^2+1\right)^2} \qquad\qquad f'\left(x\right) = \frac{1-3x^2}{\left(x^2+1\right)^3}$$

The derivative of the function will be negative for $x > \frac{1}{\sqrt{3}} = 0.5774$ and so the function will be decreasing in this range. Because the function and the series terms are the same we can also see that the series terms are decreasing for the range of *n* in our series.

Therefore, the conditions required to use the Integral Test are met! Note that it is really important to test these conditions before proceeding with the problem. It doesn't make any sense to use a test to estimate the value of a series if the test can't be used on the series. We shouldn't just assume that because we are being asked to use a test here that the test can actually be used!

Step 2
Let's start off with the partial sum using $n = 10$. This is,

$$s_{10} = \sum_{n=1}^{10} \frac{n}{\left(n^2+1\right)^2} = 0.392632317$$

Step 3
Now, to increase the accuracy of the partial sum from the previous step we know we can use each of the following two integrals.

$$\int_{10}^{\infty} \frac{x}{\left(x^2+1\right)^2}\,dx = \lim_{t\to\infty} \int_{10}^{t} \frac{x}{\left(x^2+1\right)^2}\,dx = \lim_{t\to\infty} \left[-\frac{1}{2\left(x^2+1\right)} \right]\Bigg|_{10}^{t} = \lim_{t\to\infty} \left[\frac{1}{202} - \frac{1}{2\left(t^2+1\right)} \right] = \frac{1}{202}$$

$$\int_{11}^{\infty} \frac{x}{\left(x^2+1\right)^2}\,dx = \lim_{t\to\infty} \int_{11}^{t} \frac{x}{\left(x^2+1\right)^2}\,dx = \lim_{t\to\infty} \left[-\frac{1}{2\left(x^2+1\right)} \right]\Bigg|_{11}^{t} = \lim_{t\to\infty} \left[\frac{1}{244} - \frac{1}{2\left(t^2+1\right)} \right] = \frac{1}{244}$$

Step 4
Okay, we know from the notes in this section that if *s* represents that actual value of the series that it must be in the following range.

$$0.392632317+\frac{1}{202}<s<0.392632317+\frac{1}{244}$$
$$0.397582813<s<0.396730678$$

Step 5
Finally, if we average the two numbers above we can get a better estimate of,

$$\boxed{s\approx 0.397156745}$$

---

2. Use the Comparison Test and $n=20$ to estimate the value of $\sum_{n=3}^{\infty}\frac{1}{n^3\ln(n)}$.

Step 1
Since we are being asked to use the Comparison Test to estimate the value of the series we should first make sure that the Comparison Test can actually be used on this series.

In this case that is easy enough because, for our range of *n*, the series terms are clearly positive and so we can use the Comparison Test.

Note that it is really important to test these conditions before proceeding with the problem. It doesn't make any sense to use a test to estimate the value of a series if the test can't be used on the series. We shouldn't just assume that because we are being asked to use a test here that the test can actually be used!

Step 2
Let's start off with the partial sum using $n=20$. This is,

$$s_{20}=\sum_{n=3}^{20}\frac{1}{n^3\ln(n)}=0.057315878$$

Step 3
Now, let's see if we can get can get an error estimate on this approximation of the series value. To do that we'll first need to do the Comparison Test on this series.

That is easy enough for this series once we notice that $\ln(n)$ is an increasing function and so $\ln(n)\ge\ln(3)$. Therefore, we get,

$$\frac{1}{n^3\ln(n)} \le \frac{1}{n^3\ln(3)} = \frac{1}{\ln(3)}\frac{1}{n^3}$$

Step 4
We now know, from the discussion in the notes, that an upper bound on the value of the remainder (*i.e.* the error between the approximation and exact value) is,

$$R_{20} \le T_{20} = \sum_{n=21}^{\infty}\frac{1}{n^3\ln(3)} < \int_{20}^{\infty}\frac{1}{x^3\ln(3)}\,dx$$
$$= \lim_{t\to\infty}\int_{20}^{t}\frac{1}{x^3\ln(3)}\,dx = \lim_{t\to\infty}\left(-\frac{1}{2x^2\ln(3)}\right)\Bigg|_{20}^{t}$$
$$= \lim_{t\to\infty}\left(\frac{1}{800\ln(3)} - \frac{1}{2t^2\ln(3)}\right) = \frac{1}{800\ln(3)}$$

Step 5
So, we can estimate that the value of the series is,

$$\boxed{s \approx 0.057315878}$$

and the error on this estimate will be no more than $\frac{1}{800\ln(3)} = 0.001137799$.

---

3. Use the Alternating Series Test and $n = 16$ to estimate the value of $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n n}{n^2+1}$.

Step 1
Since we are being asked to use the Alternating Series Test to estimate the value of the series we should first make sure that the Alternating Series Test can actually be used on this series.

First, note that the b_n for this series are,

$$b_n = \frac{n}{n^2+1}$$

and they are positive and with a quick derivative we can see they are decreasing and so the Alternating Series Test can be used here.

Note that it is really important to test these conditions before proceeding with the problem. It doesn't make any sense to use a test to estimate the value of a series if the test can't be used on the series. We shouldn't just assume that because we are being asked to use a test here that the test can actually be used!

Step 2
Let's start off with the partial sum using $n = 16$. This is,

$$s_{16} = \sum_{n=2}^{16} \frac{(-1)^n n}{n^2 + 1} = 0.260554530$$

Step 3
Now, we know, from the discussion in the notes, that an upper bound on the absolute value of the remainder (*i.e.* the error between the approximation and exact value) is nothing more than,

$$b_{17} = \frac{17}{290} = 0.058620690$$

Step 4
So, we can estimate that the value of the series is,

$$\boxed{s \approx 0.260554530}$$

and the error on this estimate will be no more than 0.058620690.

---

4. Use the Ratio Test and $n = 8$ to estimate the value of $\sum_{n=1}^{\infty} \frac{3^{1+n}}{n\,2^{3+2n}}$.

Step 1
First notice that the terms are positive and so we can use the Ratio Test to do the estimate. Remember that this is a requirement only to use the Ratio Test to get an estimate of the series value and is not an actual requirement to use the Ratio Test to determine if the series converges or diverges.

So, let's start off with the partial sum using $n = 8$. This is,

$$s_8 = \sum_{n=1}^{8} \frac{3^{1+n}}{n\,2^{3+2n}} = 0.509881435$$

Step 2
Now, to get an upper bound on the value of the remainder (*i.e.* the error between the approximation and exact value) we need the following ratio,

$$r_n = \frac{a_{n+1}}{a_n} = \frac{3^{2+n}}{(n+1)2^{5+2n}} \frac{n\,2^{3+2n}}{3^{1+n}} = \frac{3n}{4(n+1)}$$

We'll also potentially need the limit,

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3n}{4(n+1)} = \frac{3}{4}$$

Step 3

Next, we need to know if the r_n form an increasing or decreasing sequence. A quick application of Calculus I will answer this.

$$f(x) = \frac{3x}{4(x+1)} \qquad\qquad f'(x) = \frac{3}{4(x+1)^2} > 0$$

As noted above the derivative is always positive and so the function, and hence the r_n are increasing.

Step 4

The upper bound on the remainder is then,

$$R_8 \le \frac{a_9}{1-L} = \frac{\frac{6561}{2,097,152}}{1-\frac{3}{4}} = 0.012514114$$

Step 5

So, we can estimate that the value of the series is,

$$\boxed{s \approx 0.509881435}$$

and the error on this estimate will be no more than 0.012514114.

Section 4-14 : Power Series

1. For the following power series determine the interval and radius of convergence.

$$\sum_{n=0}^{\infty} \frac{1}{(-3)^{2+n}(n^2+1)}(4x-12)^n$$

Step 1
Okay, let's start off with the Ratio Test to get our hands on *L*.

$$L = \lim_{n\to\infty}\left| \frac{(4x-12)^{n+1}}{(-3)^{3+n}\left((n+1)^2+1\right)} \frac{(-3)^{2+n}(n^2+1)}{(4x-12)^n} \right| = \lim_{n\to\infty}\left| \frac{(4x-12)}{(-3)\left((n+1)^2+1\right)} \frac{(n^2+1)}{1} \right|$$

$$= |4x-12| \lim_{n\to\infty} \frac{(n^2+1)}{3\left((n+1)^2+1\right)} = \frac{1}{3}|4x-12|$$

Step 2
So, we know that the series will converge if,

$$\frac{1}{3}|4x-12| < 1 \quad \to \quad \frac{4}{3}|x-3| < 1 \quad \to \quad |x-3| < \frac{3}{4}$$

Step 3
So, from the previous step we see that the radius of convergence is $\boxed{R = \frac{3}{4}}$.

Step 4
Now, let's start working on the interval of convergence. Let's break up the inequality we got in Step 2.

$$-\frac{3}{4} < x-3 < \frac{3}{4} \quad \to \quad \frac{9}{4} < x < \frac{15}{4}$$

Step 5
To finalize the interval of convergence we need to check the end points of the inequality from Step 4.

$$x = \frac{9}{4} : \sum_{n=0}^{\infty} \frac{1}{(-3)^{2+n}(n^2+1)}(-3)^n = \sum_{n=0}^{\infty} \frac{1}{(-3)^2(n^2+1)} = \sum_{n=0}^{\infty} \frac{1}{9(n^2+1)}$$

$$x = \frac{15}{4} : \sum_{n=0}^{\infty} \frac{1}{(-1)^{2+n}(3)^{2+n}(n^2+1)}(3)^n = \sum_{n=0}^{\infty} \frac{1}{(-1)^{2+n}(3)^2(n^2+1)} = \sum_{n=0}^{\infty} \frac{(-1)^{2+n}}{9(n^2+1)}$$

Now, we can do a quick Comparison Test on the first series to see that it converges and we can do a quick Alternating Series Test on the second series to see that is also converges.

We'll leave it to you to verify both of these statements.

Step 6
The interval of convergence is below and for summary purposes the radius of convergence is also shown.

$$\boxed{\text{Interval} \;:\; \frac{9}{4} \le x \le \frac{15}{4}} \qquad\qquad \boxed{R = \frac{3}{4}}$$

---

2. For the following power series determine the interval and radius of convergence.

$$\sum_{n=0}^{\infty} \frac{n^{2n+1}}{4^{3n}} (2x+17)^n$$

Step 1
Okay, let's start off with the Root Test to get our hands on *L*.

$$L = \lim_{n\to\infty} \left| \frac{n^{2n+1}}{4^{3n}} (2x+17)^n \right|^{\frac{1}{n}} = \lim_{n\to\infty} \left| \frac{n^{2+\frac{1}{n}}}{4^3} (2x+17) \right| = \left|2x+17\right| \lim_{n\to\infty} \frac{n^{2+\frac{1}{n}}}{4^3}$$

Okay, we can see that , in this case, *L* will be infinite provided $x \ne -\frac{17}{2}$ and so the series will diverge for $x \ne -\frac{17}{2}$. We also know that the power series will converge for $x = \frac{17}{2}$ (this is the value of *a* for this series!).

Step 2
Therefore, we know that the interval of convergence is $\boxed{x = -\frac{17}{2}}$ and the radius of convergence is $\boxed{R = 0}$.

---

3. For the following power series determine the interval and radius of convergence.

$$\sum_{n=0}^{\infty} \frac{n+1}{(2n+1)!} (x-2)^n$$

Step 1
Okay, let's start off with the Ratio Test to get our hands on *L*.

$$L = \lim_{n\to\infty}\left|\frac{(n+2)(x-2)^{n+1}}{(2n+3)!}\frac{(2n+1)!}{(n+1)(x-2)^{n}}\right| = \lim_{n\to\infty}\left|\frac{(n+2)(x-2)}{(2n+3)(2n+2)(2n+1)!}\frac{(2n+1)!}{(n+1)}\right|$$
$$= |x-2|\lim_{n\to\infty}\frac{n+2}{(2n+3)(2n+2)(n+1)} = 0$$

Okay, we can see that , in this case, $L = 0$ for every *x*.

Step 2

Therefore, we know that the interval of convergence is $\boxed{-\infty < x < \infty}$ and the radius of convergence is $\boxed{R = \infty}$.

---

4. For the following power series determine the interval and radius of convergence.

$$\sum_{n=0}^{\infty}\frac{4^{1+2n}}{5^{n+1}}(x+3)^{n}$$

Step 1

Okay, let's start off with the Ratio Test to get our hands on *L*.

$$L = \lim_{n\to\infty}\left|\frac{4^{3+2n}(x+3)^{n+1}}{5^{n+2}}\frac{5^{n+1}}{4^{1+2n}(x+3)^{n}}\right| = \lim_{n\to\infty}\left|\frac{4^{2}(x+3)}{5}\right| = |x+3|\lim_{n\to\infty}\frac{16}{5} = \frac{16}{5}|x+3|$$

Step 2

So, we know that the series will converge if,

$$\frac{16}{5}|x+3| < 1 \qquad \to \qquad |x+3| < \frac{5}{16}$$

Step 3

So, from the previous step we see that the radius of convergence is $\boxed{R = \frac{5}{16}}$.

Step 4

Now, let's start working on the interval of convergence. Let's break up the inequality we got in Step 2.

$$-\frac{5}{16} < x+3 < \frac{5}{16} \qquad\qquad \to \qquad\qquad -\frac{53}{16} < x < -\frac{43}{16}$$

Step 5

To finalize the interval of convergence we need to check the end points of the inequality from Step 4.

$$x = -\frac{53}{16}\ :\ \sum_{n=0}^{\infty}\frac{4^1 4^{2n}}{5^n 5^1}\left(-\frac{5}{16}\right)^n = \sum_{n=0}^{\infty}\frac{4\left(16^n\right)}{5^n\left(5\right)}\frac{\left(-1\right)^n 5^n}{16^n} = \sum_{n=0}^{\infty}\frac{4\left(-1\right)^n}{5}$$

$$x = -\frac{43}{16}\ :\ \sum_{n=0}^{\infty}\frac{4^1 4^{2n}}{5^n 5^1}\left(\frac{5}{16}\right)^n = \sum_{n=0}^{\infty}\frac{4\left(16^n\right)}{5^n\left(5\right)}\frac{{}^n 5^n}{16^n} = \sum_{n=0}^{\infty}\frac{4}{5}$$

Now,

$$\lim_{n\to\infty}\frac{4\left(-1\right)^n}{5}\quad - \quad \text{Does not exist} \qquad\qquad \lim_{n\to\infty}\frac{4}{5} = \frac{4}{5}$$

Therefore, each of these two series diverge by the Divergence Test.

Step 6
The interval of convergence is below and for summary purposes the radius of convergence is also shown.

$$\boxed{\text{Interval}\ :\ -\frac{53}{16} < x < -\frac{43}{16}} \qquad\qquad \boxed{R = \frac{5}{16}}$$

---

5. For the following power series determine the interval and radius of convergence.

$$\sum_{n=1}^{\infty}\frac{6^n}{n}\left(4x-1\right)^{n-1}$$

Step 1
Okay, let's start off with the Ratio Test to get our hands on *L*.

$$L = \lim_{n\to\infty}\left|\frac{6^{n+1}\left(4x-1\right)^n}{n+1}\frac{n}{6^n\left(4x-1\right)^{n-1}}\right| = \lim_{n\to\infty}\left|\frac{6n\left(4x-1\right)}{n+1}\right| = \left|4x-1\right|\lim_{n\to\infty}\frac{6n}{n+1} = 6\left|4x-1\right|$$

Step 2
So, we know that the series will converge if,

$$6\left|4x-1\right| < 1 \qquad \to \qquad 24\left|x-\tfrac{1}{4}\right| < 1 \qquad \to \qquad \left|x-\tfrac{1}{4}\right| < \frac{1}{24}$$

Step 3
So, from the previous step we see that the radius of convergence is $\boxed{R = \tfrac{1}{24}}$.

Step 4

Now, let's start working on the interval of convergence. Let's break up the inequality we got in Step 2.

$$-\frac{1}{24} < x - \frac{1}{4} < \frac{1}{24} \qquad \to \qquad \frac{5}{24} < x < \frac{7}{24}$$

Step 5

To finalize the interval of convergence we need to check the end points of the inequality from Step 4.

$$x = \frac{5}{24} : \sum_{n=1}^{\infty} \frac{6^n}{n}\left(-\frac{1}{6}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{6^n}{n}\frac{\left(-1\right)^{n-1}}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{6\left(-1\right)^{n-1}}{n}$$

$$x = \frac{7}{24} : \sum_{n=1}^{\infty} \frac{6^n}{n}\left(\frac{1}{6}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{6^n}{n}\frac{1}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{6}{n}$$

Now, the first series is an alternating harmonic series which we know converges (or you could just do a quick Alternating Series Test to verify this) and the second series diverges by the *p*-series test.

Step 6

The interval of convergence is below and for summary purposes the radius of convergence is also shown.

$$\boxed{\text{Interval} : \frac{5}{24} \le x < \frac{7}{24}} \qquad \boxed{R = \frac{1}{24}}$$

Section 4-15 : Power Series and Functions

1. Write the following function as a power series and give the interval of convergence.

$$f\left(x\right)=\frac{6}{1+7x^4}$$

Step 1
First, in order to use the formula from this section we know that we need the numerator to be a one. That is easy enough to "fix" up as follows,

$$f\left(x\right)=6\frac{1}{1+7x^4}$$

Step 2
Next, we know we need the denominator to be in the form $1-p$ and again that is easy enough, in this case, to rewrite the denominator to get the following form of the function,

$$f\left(x\right)=6\frac{1}{1-\left(-7x^4\right)}$$

Step 3
At this point we can use the formula from the notes to write this as a power series. Doing this gives,

$$f\left(x\right)=6\frac{1}{1-\left(-7x^4\right)}=6\sum_{n=0}^{\infty}\left(-7x^4\right)^n \qquad \text{provided } \left|-7x^4\right|<1$$

Step 4
Now, recall the basic "rules" for the form of the series answer. We don't want anything out in front of the series and we want a single *x* with a single exponent on it.

These are easy enough rules to take care of. All we need to do is move whatever is in front of the series to the inside of the series and use basic exponent rules to take care of the *x* "rule". Doing all this gives,

$$f\left(x\right)=\sum_{n=0}^{\infty}6\left(-7\right)^n\left(x^4\right)^n=\sum_{n=0}^{\infty}6\left(-7\right)^n x^{4n} \qquad \text{provided } \left|-7x^4\right|<1$$

Step 5
To get the interval of convergence all we need to do is do a little work on the "provided" portion of the result from the last step to get,

$$\left|-7x^4\right|<1 \quad\to\quad 7\left|x\right|^4<1 \quad\to\quad \left|x\right|^4<\frac{1}{7} \quad\to\quad \left|x\right|<\frac{1}{7^{\frac{1}{4}}} \quad\to\quad -\frac{1}{7^{\frac{1}{4}}}<x<\frac{1}{7^{\frac{1}{4}}}$$

Note that we don't need to check the endpoints of this interval since we already know that we only get convergence with the strict inequalities and we will get divergence for everything else.

Step 6
The answers for this problem are then,

$$\boxed{\text{Power Series} \quad : \quad \frac{6}{1+7x^4} = \sum_{n=0}^{\infty} 6(-7)^n x^{4n}} \qquad \boxed{\text{Interval} \quad : \quad -\frac{1}{7^{\frac{1}{4}}} < x < \frac{1}{7^{\frac{1}{4}}}}$$

---

2. Write the following function as a power series and give the interval of convergence.

$$f(x) = \frac{x^3}{3-x^2}$$

Step 1
First, in order to use the formula from this section we know that we need the numerator to be a one. That is easy enough to "fix" up as follows,

$$f(x) = x^3 \frac{1}{3-x^2}$$

Step 2
Next, we know we need the denominator to be in the form $1-p$ and again that is easy enough, in this case, to rewrite the denominator by factoring a 3 out of the denominator as follows,

$$f(x) = \frac{x^3}{3} \frac{1}{1-\frac{1}{3}x^2}$$

Step 3
At this point we can use the formula from the notes to write this as a power series. Doing this gives,

$$f(x) = \frac{x^3}{3} \frac{1}{1-\frac{1}{3}x^2} = \frac{x^3}{3} \sum_{n=0}^{\infty} \left(\tfrac{1}{3}x^2\right)^n \qquad \text{provided } \left|\tfrac{1}{3}x^2\right| < 1$$

Step 4
Now, recall the basic "rules" for the form of the series answer. We don't want anything out in front of the series and we want a single *x* with a single exponent on it.

These are easy enough rules to take care of. All we need to do is move whatever is in front of the series to the inside of the series and use basic exponent rules to take care of the *x* "rule". Doing all this gives,

$$f(x) = \frac{x^3}{3}\sum_{n=0}^{\infty}\left(\tfrac{1}{3}x^2\right)^n = \sum_{n=0}^{\infty}\tfrac{1}{3}x^3\left(\tfrac{1}{3}\right)^n\left(x^2\right)^n = \sum_{n=0}^{\infty}\left(\tfrac{1}{3}\right)^{n+1}x^{2n+3} \qquad \text{provided } \left|\tfrac{1}{3}x^2\right| < 1$$

Step 5
To get the interval of convergence all we need to do is do a little work on the "provided" portion of the result from the last step to get,

$$\left|\tfrac{1}{3}x^2\right| < 1 \quad \to \quad \tfrac{1}{3}|x|^2 < 1 \quad \to \quad |x|^2 < 3 \quad \to \quad |x| < \sqrt{3} \quad \to \quad -\sqrt{3} < x < \sqrt{3}$$

Note that we don't need to check the endpoints of this interval since we already know that we only get convergence with the strict inequalities and we will get divergence for everything else.

Step 6
The answers for this problem are then,

$$\boxed{\text{Power Series} \;:\; \frac{x^3}{3-x^2} = \sum_{n=0}^{\infty}\left(\tfrac{1}{3}\right)^{n+1}x^{2n+3}} \qquad \boxed{\text{Interval} \;:\; -\sqrt{3} < x < \sqrt{3}}$$

---

3. Write the following function as a power series and give the interval of convergence.

$$f(x) = \frac{3x^2}{5 - 2\sqrt[3]{x}}$$

Step 1
First, in order to use the formula from this section we know that we need the numerator to be a one. That is easy enough to "fix" up as follows,

$$f(x) = 3x^2\frac{1}{5 - 2\sqrt[3]{x}}$$

Step 2
Next, we know we need the denominator to be in the form $1 - p$ and again that is easy enough, in this case, to rewrite the denominator by factoring a 5 out of the denominator as follows,

$$f(x) = \frac{3x^2}{5}\frac{1}{1 - \frac{2}{5}\sqrt[3]{x}}$$

Step 3
At this point we can use the formula from the notes to write this as a power series. Doing this gives,

$$f\left(x\right)=\frac{3x^2}{5}\frac{1}{1-\frac{2}{5}\sqrt[3]{x}}=\frac{3x^2}{5}\sum_{n=0}^{\infty}\left(\tfrac{2}{5}\sqrt[3]{x}\right)^n \qquad \text{provided } \left|\tfrac{2}{5}\sqrt[3]{x}\right|<1$$

Step 4
Now, recall the basic "rules" for the form of the series answer. We don't want anything out in front of the series and we want a single *x* with a single exponent on it.

These are easy enough rules to take care of. All we need to do is move whatever is in front of the series to the inside of the series and use basic exponent rules to take care of the *x* "rule". Doing all this gives,

$$f\left(x\right)=\frac{3x^2}{5}\sum_{n=0}^{\infty}\left(\tfrac{2}{5}\sqrt[3]{x}\right)^n=\sum_{n=0}^{\infty}\tfrac{3}{5}x^2\left(\tfrac{2}{5}\right)^n\left(x^{\frac{1}{3}}\right)^n=\sum_{n=0}^{\infty}\tfrac{3}{5}\left(\tfrac{2}{5}\right)^n x^{\frac{1}{3}n+2} \qquad \text{provided } \left|\tfrac{2}{5}\sqrt[3]{x}\right|<1$$

Step 5
To get the interval of convergence all we need to do is do a little work on the "provided" portion of the result from the last step to get,

$$\left|\tfrac{2}{5}\sqrt[3]{x}\right|<1 \quad\to\quad \tfrac{2}{5}\left|x\right|^{\frac{1}{3}}<1 \quad\to\quad \left|x\right|^{\frac{1}{3}}<\tfrac{5}{2} \quad\to\quad \left|x\right|<\tfrac{125}{8} \quad\to\quad -\tfrac{125}{8}<x<\tfrac{125}{8}$$

Note that we don't need to check the endpoints of this interval since we already know that we only get convergence with the strict inequalities and we will get divergence for everything else.

Step 6
The answers for this problem are then,

$$\boxed{\text{Power Series} \ : \ \frac{3x^2}{5-2\sqrt[3]{x}}=\sum_{n=0}^{\infty}\tfrac{3}{5}\left(\tfrac{2}{5}\right)^n x^{\frac{1}{3}n+2}} \qquad \boxed{\text{Interval} \ : \ -\tfrac{125}{8}<x<\tfrac{125}{8}}$$

---

4. Give a power series representation for the derivative of the following function.

$$g\left(x\right)=\frac{5x}{1-3x^5}$$

Hint : While we could differentiate the function and then attempt to find a power series representation that seems like a lot of work. It's a good think that we know how to differentiate power series.

Step 1
First let's notice that we can quickly find a power series representation for this function. Here is that work.

$$g\left(x\right)=5x\frac{1}{1-3x^5}=5x\sum_{n=0}^{\infty}\left(3x^5\right)^n=\sum_{n=0}^{\infty}5x\left(3^n\right)x^{5n}=\sum_{n=0}^{\infty}5\left(3^n\right)x^{5n+1}$$

Step 2
Now, we know how to differentiate power series and we know that the derivative of the power series representation of a function is the power series representation of the derivative of the function.

Therefore,

$$g'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} 5\left(3^n\right)x^{5n+1}\right] = \boxed{\sum_{n=0}^{\infty} 5\left(5n+1\right)\left(3^n\right)x^{5n}}$$

Remember that to differentiate a power series all we need to do is differentiate the term of the power series with respect to *x*.

---

5. Give a power series representation for the integral of the following function.

$$h(x) = \frac{x^4}{9+x^2}$$

Hint : Integrating this function seems like (potentially) a lot of work, not to mention determining a power series representation of the result. It's a good think that we know how to integrate power series.

Step 1
First let's notice that we can quickly find a power series representation for this function. Here is that work.

$$h(x) = \frac{x^4}{9}\frac{1}{1-\left(-\frac{1}{9}x^2\right)} = \frac{x^4}{9}\sum_{n=0}^{\infty}\left(-\tfrac{1}{9}x^2\right)^n = \sum_{n=0}^{\infty}\tfrac{1}{9}x^4\left(-1\right)^n\left(\tfrac{1}{9}\right)^n x^{2n} = \sum_{n=0}^{\infty}\left(-1\right)^n\left(\tfrac{1}{9}\right)^{n+1}x^{2n+4}$$

Step 2
Now, we know how to integrate power series and we know that the integral of the power series representation of a function is the power series representation of the integral of the function.

Therefore,

$$\int h(x)\,dx = \int\sum_{n=0}^{\infty}\left(-1\right)^n\left(\tfrac{1}{9}\right)^{n+1}x^{2n+4}\,dx = \boxed{c + \sum_{n=0}^{\infty}\tfrac{1}{2n+5}\left(-1\right)^n\left(\tfrac{1}{9}\right)^{n+1}x^{2n+5}}$$

Remember that to integrate a power series all we need to do is integrate the term of the power series and we can't forget to add on the "+c" since we're doing an indefinite integral.

Section 4-16 : Taylor Series

1. Use one of the Taylor Series derived in the notes to determine the Taylor Series for $f\left(x\right)=\cos\left(4x\right)$ about $x=0$.

Step 1
There really isn't all that much to do here for this problem. We are working with cosine and want the Taylor series about $x=0$ and so we can use the Taylor series for cosine derived in the notes to get,

$$\cos\left(4x\right)=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}\left(4x\right)^{2n}}{\left(2n\right)!}$$

Step 2
Now, recall the basic "rules" for the form of the series answer. We don't want anything out in front of the series and we want a single *x* with a single exponent on it.

In this case we don't have anything out in front of the series to worry about so all we need to do is use the basic exponent rules on the 2*x* term to get,

$$\cos\left(4x\right)=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}4^{2n}x^{2n}}{\left(2n\right)!}=\boxed{\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}16^{n}x^{2n}}{\left(2n\right)!}}$$

2. Use one of the Taylor Series derived in the notes to determine the Taylor Series for $f\left(x\right)=x^{6}\mathbf{e}^{2x^{3}}$ about $x=0$.

Step 1
There really isn't all that much to do here for this problem. We are working with the exponential function and want the Taylor series about $x=0$ and so we can use the Taylor series for the exponential function derived in the notes to get,

$$x^{6}\mathbf{e}^{2x^{3}}=x^{6}\sum_{n=0}^{\infty}\frac{\left(2x^{3}\right)^{n}}{n!}$$

Note that we only convert the exponential using the Taylor series derived in the notes and, at this point, we just leave the x^{6} alone in front of the series.

Step 2

Now, recall the basic "rules" for the form of the series answer. We don't want anything out in front of the series and we want a single *x* with a single exponent on it.

These are easy enough rules to take care of. All we need to do is move whatever is in front of the series to the inside of the series and use basic exponent rules to take care of the *x* "rule". Doing all this gives,

$$x^6\mathbf{e}^{2x^3} = x^6\sum_{n=0}^{\infty}\frac{\left(2x^3\right)^n}{n!} = \sum_{n=0}^{\infty}x^6\,\frac{2^n\left(x^3\right)^n}{n!} = \boxed{\sum_{n=0}^{\infty}\frac{2^n x^{3n+6}}{n!}}$$

---

3. Find the Taylor Series for $f\left(x\right)=\mathbf{e}^{-6x}$ about $x=-4$.

Step 1

Because we are working about $x=-4$ in this problem we are not able to just use the formula derived in class for the exponential function because that requires us to be working about $x=0$.

Step 2

So, we'll need to start over from the beginning and start taking some derivatives of the function.

$$\begin{aligned}
n=0: &\quad f\left(x\right)=\mathbf{e}^{-6x}\\
n=1: &\quad f'\left(x\right)=-6\mathbf{e}^{-6x}\\
n=2: &\quad f''\left(x\right)=\left(-6\right)^2\mathbf{e}^{-6x}\\
n=3: &\quad f^{(3)}\left(x\right)=\left(-6\right)^3\mathbf{e}^{-6x}\\
n=4: &\quad f^{(4)}\left(x\right)=\left(-6\right)^4\mathbf{e}^{-6x}
\end{aligned}$$

Remember that, in general, we're going to need to go out to at least $n=4$ for most of these problems to make sure that we can get the formula for the general term in the Taylor Series.

Also, remember to NOT multiply things out when taking derivatives! Doing that will make your life much harder when it comes time to find the general formula.

Step 3

It is now time to see if we can get a formula for the general term in the Taylor Series.

In this case, it is (hopefully) pretty simple to catch the pattern in the derivatives above. The general term is given by,

$$f^{(n)}\left(x\right)=\left(-6\right)^n\mathbf{e}^{-6x}\qquad n=0,1,2,3,\ldots$$

As noted this formula works all the way back to $n=0$. It is important to make sure that you check this formula to determine just how far back it will work. We will, on occasion, get formulas that will not work for the first couple of *n*'s and we need to know that before we start writing down the Taylor Series.

Step 4
Now, recall that we don't really want the general term at any *x*. We want the general term at $x=-4$. This is,

$$f^{(n)}\left(-4\right)=\left(-6\right)^n\mathbf{e}^{24}\qquad n=0,1,2,3,\ldots$$

Step 5
Okay, at this point we can formally write down the Taylor Series for this problem.

$$\mathbf{e}^{-6x}=\sum_{n=0}^{\infty}\frac{f^{(n)}\left(-4\right)}{n!}\left(x+4\right)^n=\boxed{\sum_{n=0}^{\infty}\frac{\left(-6\right)^n\mathbf{e}^{24}}{n!}\left(x+4\right)^n}$$

---

4. Find the Taylor Series for $f\left(x\right)=\ln\left(3+4x\right)$ about $x=0$.

Step 1
Okay, we'll need to start off this problem by taking a few derivatives of the function.

$$\begin{aligned}
n=0:&\qquad f\left(x\right)=\ln\left(3+4x\right)\\
n=1:&\qquad f'\left(x\right)=\frac{4}{3+4x}=4\left(3+4x\right)^{-1}\\
n=2:&\qquad f''\left(x\right)=-4^2\left(3+4x\right)^{-2}\\
n=3:&\qquad f^{(3)}\left(x\right)=4^3\left(2\right)\left(3+4x\right)^{-3}\\
n=4:&\qquad f^{(4)}\left(x\right)=-4^4\left(2\right)\left(3\right)\left(3+4x\right)^{-4}\\
n=5:&\qquad f^{(5)}\left(x\right)=4^5\left(2\right)\left(3\right)\left(4\right)\left(3+4x\right)^{-5}
\end{aligned}$$

Remember that, in general, we're going to need to go out to at least $n=4$ for most of these problems to make sure that we can get the formula for the general term in the Taylor Series.

Also, remember to NOT multiply things out when taking derivatives! Doing that will make your life much harder when it comes time to find the general formula. In this case we "merged" all the 4's that came from the chain rule into a single term but left it as an exponent rather than get an actual value. This is not uncommon with these kinds of problems. The exponents we dropped down for the derivatives we left alone with the exception of dealing with the signs.

Step 2

It is now time to see if we can get a formula for the general term in the Taylor Series.

Hopefully you can see the pattern in the derivatives above. The general term is given by,

$$
\begin{aligned}
f^{(0)}(x) &= \ln(3+4x) && n=0 \\
f^{(n)}(x) &= (-1)^{n+1}4^n(n-1)!(3+4x)^{-n} && n=1,2,3,\ldots
\end{aligned}
$$

As noted this formula works all the way back to $n=1$ but clearly does not work for $n=0$. It is problems like this one that make it clear why we always need to check our proposed formula for the general solution to see just how far back it works.

Step 3

Now, recall that we don't really want the general term at any *x*. We want the general term at $x=0$. This is,

$$
\begin{aligned}
f^{(0)}(0) &= \ln(3) && n=0 \\
f^{(n)}(0) &= (-1)^{n+1}4^n(n-1)!(3)^{-n} \\
&= (-1)^{n+1}4^n(n-1)!\frac{1}{3^n} \\
&= (-1)^{n+1}\left(\frac{4}{3}\right)^n(n-1)! && n=1,2,3,\ldots
\end{aligned}
$$

We did a little simplification for the second one just to make it a little simpler.

Step 4

Okay, at this point we can formally write down the Taylor Series for this problem. However, before we actually do that recall that our general term formula did not work for $n=0$ and so we'll need to first strip that out of the series before we put the general formula in.

$$
\begin{aligned}
\ln(3+4x) &= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \\
&= f(0)+\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \\
&= \ln(3)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(\frac{4}{3}\right)^n(n-1)!}{n!}x^n \\
&= \boxed{\ln(3)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(\frac{4}{3}\right)^n}{n}x^n}
\end{aligned}
$$

Don't forget to simplify/cancel where we can in the final answer. In this case we could do some simplifying with the factorials.

5. Find the Taylor Series for $f\left(x\right)=\frac{7}{x^4}$ about $x=-3$.

Step 1
Okay, we'll need to start off this problem by taking a few derivatives of the function.

$$\begin{aligned}
n=0: &\quad f\left(x\right)=\frac{7}{x^4}=7x^{-4}\\
n=1: &\quad f'\left(x\right)=-7\left(4\right)x^{-5}\\
n=2: &\quad f''\left(x\right)=7\left(4\right)\left(5\right)x^{-6}\\
n=3: &\quad f^{(3)}\left(x\right)=-7\left(4\right)\left(5\right)\left(6\right)x^{-7}\\
n=4: &\quad f^{(4)}\left(x\right)=7\left(4\right)\left(5\right)\left(6\right)\left(7\right)x^{-8}
\end{aligned}$$

Remember that, in general, we're going to need to go out to at least $n=4$ for most of these problems to make sure that we can get the formula for the general term in the Taylor Series.

Also, remember to NOT multiply things out when taking derivatives! Doing that will make your life much harder when it comes time to find the general formula. In this case the only "simplification" we did was to multiply out the minus signs that came from the exponents upon doing the derivatives. That is a fairly common thing to do with these kinds of problems.

Step 2
It is now time to see if we can get a formula for the general term in the Taylor Series.

Hopefully you can see the pattern in the derivatives above. The general term is given by,

$$\begin{aligned}
f^{(n)}\left(x\right)&=7\left(-1\right)^n\frac{\left(2\right)\left(3\right)}{\left(2\right)\left(3\right)}\left(4\right)\left(5\right)\left(6\right)\cdots\left(n+3\right)x^{-8}\\
&=7\left(-1\right)^n\frac{\left(2\right)\left(3\right)\left(4\right)\left(5\right)\left(6\right)\cdots\left(n+3\right)}{6}x^{-8}\\
&=\frac{7}{6}\left(-1\right)^n\left(n+3\right)!\,x^{-(n+4)} \qquad\qquad n=0,1,2,3,\ldots
\end{aligned}$$

This formula may have been a little trickier to get. We almost had a factorial in the derivatives but each one was missing the $\left(2\right)\left(3\right)$ part that would be needed to get the factorial to show up. Because that was all that was missing and it was missing in each of the derivatives we multiplied each derivative by

$\frac{(2)(3)}{(2)(3)}$ (*i.e.* a really fancy way of writing one…). We could then use the numerator of this to complete the factorial and the denominator was just left alone.

Also, as noted this formula works all the way back to $n = 0$. It is important to make sure that you check this formula to determine just how far back it will work. We will, on occasion, get formulas that will not work for the first couple of *n*'s and we need to know that before we start writing down the Taylor Series.

Step 3
Now, recall that we don't really want the general term at any *x*. We want the general term at $x = -3$. This is,

$$\begin{aligned} f^{(n)}(-3) &= \frac{7}{6}(-1)^n (n+3)!(-3)^{-(n+4)} \\ &= \frac{7(-1)^n (n+3)!}{6(-3)^{n+4}} \\ &= \frac{7(-1)^n (n+3)!}{6(-1)^{n+4}(3)^{n+4}} \\ &= \frac{7(n+3)!}{6(-1)^4 (3)^{n+4}} \\ &= \frac{7(n+3)!}{6(3)^{n+4}} \qquad n = 1, 2, 3, \ldots \end{aligned}$$

We did a little simplification here so we could cancel out all the alternating signs that were present in the term.

Step 4
Okay, at this point we can formally write down the Taylor Series for this problem.

$$\frac{7}{x^4} = \sum_{n=0}^{\infty} \frac{f^{(n)}(-3)}{n!}(x+3)^n = \sum_{n=0}^{\infty} \frac{7(n+3)!}{6(3)^{n+4} n!}(x+3)^n = \boxed{\sum_{n=0}^{\infty} \frac{7(n+3)(n+2)(n+1)}{6(3)^{n+4}}(x+3)^n}$$

Don't forget to simplify/cancel where we can in the final answer. In this case we could do some simplifying with the factorials.

---

6. Find the Taylor Series for $f(x) = 7x^2 - 6x + 1$ about $x = 2$.

Step 1

First, let's not get too excited about the fact that we have a polynomial here for this problem. It works exactly the same way with a few small differences.

We'll start off by taking a few derivatives of the function and evaluating them at $x = 2$

$$\begin{array}{lll} n=0: & f(x) = 7x^2 - 6x + 1 & f(2) = 17 \\ n=1: & f'(x) = 14x - 6 & f'(2) = 22 \\ n=2: & f''(x) = 14 & f''(2) = 14 \\ n \geq 3: & f^{(n)}(x) = 0 & f^{(n)}(2) = 0 \end{array}$$

Okay, this is where one of the differences between a polynomial and the other types of functions we typically see with Taylor Series problems. After some point all the derivatives will be zero. That is not something to get excited about. In fact, it actually makes the problem a little easier!

Because all the derivatives are zero after some point we don't need a formula for the general term. All we need are the values of the non-zero derivative terms.

Step 2
Once we have the values from the previous step all we need to do is write down the Taylor Series. To do that all we need to do is strip all the non-zero terms from the series and then acknowledge that the remainder will just be zero (all the remaining terms are zero after all!).

Doing this gives,

$$\begin{aligned} 7x^2 - 6x + 1 &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\ &= f(2) + f'(2)(x-2) + \tfrac{1}{2} f''(2)(x-2)^2 + \sum_{n=3}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\ &= \boxed{17 + 22(x-2) + 7(x-2)^2} \end{aligned}$$

It looks a little strange but there it is. Do not multiply/simplify this out. This really is the answer we are looking for.

Also, don't think that this is a problem that is just done to make you work another problem. There are applications of series (beyond the scope of this course however…) that really do require this kind of thing to be done as strange as that might sound!

Section 4-17 : Applications of Series

1. Determine a Taylor Series about $x = 0$ for the following integral.

$$\int \frac{\mathbf{e}^x - 1}{x}\,dx$$

Step 1
This problem isn't quite as hard as it might first appear. We know how to integrate a series so all we really need to do here is find a Taylor series for the integrand and then integrate that.

Step 2
Okay, let's start out by noting that we are working about $x = 0$ and that means we can use the formula for the Taylor Series of the exponential function. For reference purposes this is,

$$\mathbf{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Next, let's strip out the $n = 0$ term from this and then subtract one. Doing this gives,

$$\mathbf{e}^x - 1 = \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}\right] - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

Of course, in doing the above step all we really managed to do was eliminate the $n = 0$ term from the series. In fact, that was not a bad thing to have happened as well see shortly.

Finally, let's divide the whole thing by *x*. This gives,

$$\frac{\mathbf{e}^x - 1}{x} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

We moved the *x* that was outside the series into the series. This is required in order to do the integral of the series. We only want a single *x* in the problem and we now have that.

Also note that while the function on the left has a division by zero issue the series on the right does not have this problem. All of the *x*'s in the series have positive or zero exponents! This is a really good thing.

Of course, the other good thing that we have at this point is that we've managed to find a series representation for the integrand!

Step 3
All we need to do now is compute the integral of the series to get a series representation of the integral.

$$\int \frac{\mathbf{e}^x - 1}{x}\,dx = \int \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}\,dx = \boxed{c + \sum_{n=1}^{\infty} \frac{x^n}{(n)(n!)}}$$

---

2. Write down $T_2(x)$, $T_3(x)$ and $T_4(x)$ for the Taylor Series of $f(x) = \mathbf{e}^{-6x}$ about $x = -4$. Graph all three of the Taylor polynomials and $f(x)$ on the same graph for the interval $[-8, -2]$.

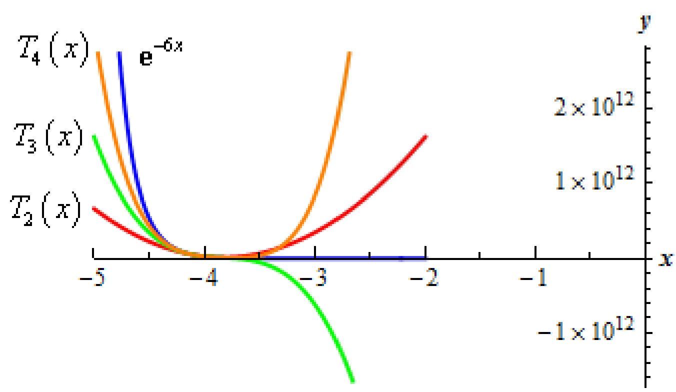
Step 1
The first thing we need to do here is get the Taylor Series for $f(x) = \mathbf{e}^{-6x}$ about $x = -4$. Luckily enough for us we did that in Problem 3 of the previous section. Here is the Taylor Series we derived in that problem.

$$\mathbf{e}^{-6x} = \sum_{n=0}^{\infty} \frac{(-6)^n \mathbf{e}^{24}}{n!}(x+4)^n$$

Step 2
Here are the three Taylor polynomials needed for this problem.

$$T_2(x) = \mathbf{e}^{24} - 6\mathbf{e}^{24}(x+4) + 18\mathbf{e}^{24}(x+4)^2$$
$$T_3(x) = \mathbf{e}^{24} - 6\mathbf{e}^{24}(x+4) + 18\mathbf{e}^{24}(x+4)^2 - 36\mathbf{e}^{24}(x+4)^3$$
$$T_4(x) = \mathbf{e}^{24} - 6\mathbf{e}^{24}(x+4) + 18\mathbf{e}^{24}(x+4)^2 - 36\mathbf{e}^{24}(x+4)^3 + 54\mathbf{e}^{24}(x+4)^4$$

Step 3
Here is the graph for this problem.



We can see that as long as we stay "near" $x = -4$ the graphs of the polynomial are pretty close to the graph of the exponential function. However, if we get too far away the graphs really do start to diverge from the graph of the exponential function.

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3. Write down $T_3(x)$, $T_4(x)$ and $T_5(x)$ for the Taylor Series of $f(x)=\ln(3+4x)$ about $x=0$. Graph all three of the Taylor polynomials and $f(x)$ on the same graph for the interval $\left[-\frac{1}{2},2\right]$.

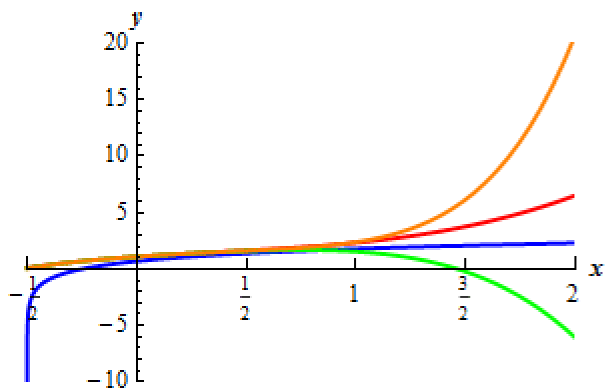
Step 1
The first thing we need to do here is get the Taylor Series for $f(x)=\ln(3+4x)$ about $x=0$. Luckily enough for us we did that in Problem 4 of the previous section. Here is the Taylor Series we derived in that problem.

$$\ln(3+4x)=\ln(3)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(\frac{4}{3}\right)^n}{n}x^n$$

Step 2
Here are the three Taylor polynomials needed for this problem.

$$T_3(x)=\ln(3)+\tfrac{4}{3}x-\tfrac{8}{9}x^2+\tfrac{64}{81}x^3$$
$$T_4(x)=\ln(3)+\tfrac{4}{3}x-\tfrac{8}{9}x^2+\tfrac{64}{81}x^3-\tfrac{64}{81}x^4$$
$$T_5(x)=\ln(3)+\tfrac{4}{3}x-\tfrac{8}{9}x^2+\tfrac{64}{81}x^3-\tfrac{64}{81}x^4+\tfrac{1024}{1215}x^5$$

Step 3
Here is the graph for this problem.



We can see that as long as we stay "near" $x=0$ the graphs of the polynomial are pretty close to the graph of the exponential function. However, if we get too far away the graphs really do start to diverge from the graph of the exponential function.

Section 4-18 : Binomial Series

1. Use the Binomial Theorem to expand $\left(4+3x\right)^5$.

Solution
Not really a lot to do with this problem. All we need to do is use the formula from the Binomial Theorem to do the expansion. Here is that work.

$$\begin{aligned}\left(4+3x\right)^5 &= \sum_{i=0}^{5}\binom{5}{i}4^{5-i}\left(3x\right)^i\\ &= \binom{5}{0}\left(4^5\right)+\binom{5}{1}\left(4^4\right)\left(3x\right)^1+\binom{5}{2}\left(4^3\right)\left(3x\right)^2+\binom{5}{3}\left(4^2\right)\left(3x\right)^3+\binom{5}{4}\left(4^1\right)\left(3x\right)^4\\ &\quad +\binom{5}{5}\left(3x\right)^5\\ &= 4^5+\left(5\right)\left(4^4\right)\left(3x\right)+\frac{5\left(4\right)}{2!}\left(4^3\right)\left(3x\right)^2+\frac{5\left(4\right)\left(3\right)}{3!}\left(4^2\right)\left(3x\right)^3+\left(5\right)\left(4\right)\left(3x\right)^4+\left(3x\right)^5\\ &= \boxed{1024+3840x+5760x^2+4320x^3+1620x^4+243x^5}\end{aligned}$$

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2. Use the Binomial Theorem to expand $\left(9-x\right)^4$.

Solution
Not really a lot to do with this problem. All we need to do is use the formula from the Binomial Theorem to do the expansion. Here is that work.

$$\begin{aligned}\left(9-x\right)^4 &= \sum_{i=0}^{4}\binom{4}{i}9^{4-i}\left(-x\right)^i\\ &= \binom{4}{0}\left(9^4\right)+\binom{4}{1}\left(9^3\right)\left(-x\right)^1+\binom{4}{2}\left(9^2\right)\left(-x\right)^2+\binom{4}{3}\left(9^1\right)\left(-x\right)^3+\binom{4}{4}\left(-x\right)^4\\ &= 9^4+\left(4\right)\left(9^3\right)\left(-x\right)+\frac{4\left(3\right)}{2!}\left(9^2\right)\left(-x\right)^2+\left(4\right)\left(9^1\right)\left(-x\right)^3+\left(-x\right)^4\\ &= \boxed{6561-2916x+486x^2-36x^3+x^4}\end{aligned}$$

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3. Write down the first four terms in the binomial series for $\left(1+3x\right)^{-6}$.

Step 1
First, we need to make sure it is in the proper form to use the Binomial Series from the notes which in this case we are already in the proper form with $k=-6$.

Step 2
Now all we need to do is plug into the formula from the notes and write down the first four terms.

$$\begin{aligned}\left(1+3x\right)^{-6} &= \sum_{i=0}^{\infty}\binom{-6}{i}\left(3x\right)^i \\ &= 1+\left(-6\right)\left(3x\right)^1+\frac{\left(-6\right)\left(-7\right)}{2!}\left(3x\right)^2+\frac{\left(-6\right)\left(-7\right)\left(-8\right)}{3!}\left(3x\right)^3+\cdots \\ &= \boxed{1-18x+189x^2-1512x^3+\cdots}\end{aligned}$$

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4. Write down the first four terms in the binomial series for $\sqrt[3]{8-2x}$.

Step 1
First, we need to make sure it is in the proper form to use the Binomial Series. Here is the proper form for this function,

$$\sqrt[3]{8-2x}=\left(8\left(1-\tfrac{1}{4}x\right)\right)^{\frac{1}{3}}=\left(8\right)^{\frac{1}{3}}\left(1-\tfrac{1}{4}x\right)^{\frac{1}{3}}=2\left(1+\left(-\tfrac{1}{4}x\right)\right)^{\frac{1}{3}}$$

Recall that for proper from we need it to be in the form "1+" and so we needed to factor the 8 out of the root and "move" the minus sign into the second term. Also, as we can see we will have $k=\frac{1}{3}$

Step 2
Now all we need to do is plug into the formula from the notes and write down the first four terms.

$$\begin{aligned}\sqrt[3]{8-2x} &= 2\left(1+\left(-\tfrac{1}{4}x\right)\right)^{\frac{1}{3}} \\ &= 2\sum_{i=0}^{\infty}\binom{\frac{1}{3}}{i}\left(-\tfrac{1}{4}x\right)^i \\ &= 2\left[1+\left(\tfrac{1}{3}\right)\left(-\tfrac{1}{4}x\right)^1+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}\left(-\tfrac{1}{4}x\right)^2+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(-\tfrac{1}{4}x\right)^3+\cdots\right] \\ &= \boxed{2-\tfrac{1}{6}x-\tfrac{1}{72}x^2-\tfrac{5}{2592}x^3+\cdots}\end{aligned}$$