CALCULUS II

Practice Problems
Series & Sequences

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Chapter 4 : Series & Sequences

Here are a set of practice problems for the Series and Sequences chapter of the Calculus II notes. If you are viewing the pdf version of this document (as opposed to viewing it on the web) this document contains only the problems themselves and no solutions are included in this document. Solutions can be found in a couple of places on the site.

1. If you'd like a pdf document containing the solutions the download tab on the website contains links to pdf's containing the solutions for the full book, chapter and section. At this time, I do not offer pdf's for solutions to individual problems.

2. If you'd like to view the solutions on the web go to the problem set web page, click the solution link for any problem and it will take you to the solution to that problem.

Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Here is a list of all the sections for which practice problems have been written as well as a brief description of the material covered in the notes for that particular section.

Sequences – In this section we define just what we mean by sequence in a math class and give the basic notation we will use with them. We will focus on the basic terminology, limits of sequences and convergence of sequences in this section. We will also give many of the basic facts and properties we'll need as we work with sequences.

More on Sequences – In this section we will continued examining sequences. We will determine if a sequence in an increasing sequence or a decreasing sequence and hence if it is a monotonic sequence. We will also determine a sequence is bounded below, bounded above and/or bounded.

Series – The Basics – In this section we will formally define an infinite series. We will also give many of the basic facts, properties and ways we can use to manipulate a series. We will also briefly discuss how to determine if an infinite series will converge or diverge (a more in depth discussion of this topic will occur in the next section).

Convergence/Divergence of Series – In this section we will discuss in greater detail the convergence and divergence of infinite series. We will illustrate how partial sums are used to determine if an infinite series converges or diverges. We will also give the Divergence Test for series in this section.

Special Series – In this section we will look at three series that either show up regularly or have some nice properties that we wish to discuss. We will examine Geometric Series, Telescoping Series, and Harmonic Series.

Integral Test – In this section we will discuss using the Integral Test to determine if an infinite series converges or diverges. The Integral Test can be used on a infinite series provided the terms of the series are positive and decreasing. A proof of the Integral Test is also given.

[Comparison Test/Limit Comparison Test]() – In this section we will discuss using the Comparison Test and Limit Comparison Tests to determine if an infinite series converges or diverges. In order to use either test the terms of the infinite series must be positive. Proofs for both tests are also given.

[Alternating Series Test]() – In this section we will discuss using the Alternating Series Test to determine if an infinite series converges or diverges. The Alternating Series Test can be used only if the terms of the series alternate in sign. A proof of the Alternating Series Test is also given.

[Absolute Convergence]() – In this section we will have a brief discussion on absolute convergence and conditionally convergent and how they relate to convergence of infinite series.

[Ratio Test]() – In this section we will discuss using the Ratio Test to determine if an infinite series converges absolutely or diverges. The Ratio Test can be used on any series, but unfortunately will not always yield a conclusive answer as to whether a series will converge absolutely or diverge. A proof of the Ratio Test is also given.

[Root Test]() – In this section we will discuss using the Root Test to determine if an infinite series converges absolutely or diverges. The Root Test can be used on any series, but unfortunately will not always yield a conclusive answer as to whether a series will converge absolutely or diverge. A proof of the Root Test is also given.

[Strategy for Series]() – In this section we give a general set of guidelines for determining which test to use in determining if an infinite series will converge or diverge. Note as well that there really isn't one set of guidelines that will always work and so you always need to be flexible in following this set of guidelines. A summary of all the various tests, as well as conditions that must be met to use them, we discussed in this chapter are also given in this section.

[Estimating the Value of a Series]() – In this section we will discuss how the Integral Test, Comparison Test, Alternating Series Test and the Ratio Test can, on occasion, be used to estimating the value of an infinite series.

[Power Series]() – In this section we will give the definition of the power series as well as the definition of the radius of convergence and interval of convergence for a power series. We will also illustrate how the Ratio Test and Root Test can be used to determine the radius and interval of convergence for a power series.

[Power Series and Functions]() – In this section we discuss how the formula for a convergent Geometric Series can be used to represent some functions as power series. To use the Geometric Series formula, the function must be able to be put into a specific form, which is often impossible. However, use of this formula does quickly illustrate how functions can be represented as a power series. We also discuss differentiation and integration of power series.

[Taylor Series]() – In this section we will discuss how to find the Taylor/Maclaurin Series for a function. This will work for a much wider variety of function than the method discussed in the previous section at the expense of some often unpleasant work. We also derive some well known formulas for Taylor series of $\mathbf{e}^x$, $\cos(x)$ and $\sin(x)$ around $x=0$.

Applications of Series – In this section we will take a quick look at a couple of applications of series. We will illustrate how we can find a series representation for indefinite integrals that cannot be evaluated by any other method. We will also see how we can use the first few terms of a power series to approximate a function.

Binomial Series – In this section we will give the Binomial Theorem and illustrate how it can be used to quickly expand terms in the form $(a+b)^n$ when n is an integer. In addition, when n is not an integer an extension to the Binomial Theorem can be used to give a power series representation of the term.

Section 4-1 : Sequences

For problems 1 & 2 list the first 5 terms of the sequence.

1. $\left\{ \dfrac{4n}{n^2 - 7} \right\}_{n=0}^{\infty}$

2. $\left\{ \dfrac{\left(-1\right)^{n+1}}{2n + \left(-3\right)^n} \right\}_{n=2}^{\infty}$

For problems 3 – 6 determine if the given sequence converges or diverges. If it converges what is its limit?

3. $\left\{ \dfrac{n^2 - 7n + 3}{1 + 10n - 4n^2} \right\}_{n=3}^{\infty}$

4. $\left\{ \dfrac{\left(-1\right)^{n-2} n^2}{4 + n^3} \right\}_{n=0}^{\infty}$

5. $\left\{ \dfrac{\mathbf{e}^{5n}}{3 - \mathbf{e}^{2n}} \right\}_{n=1}^{\infty}$

6. $\left\{ \dfrac{\ln\left(n+2\right)}{\ln\left(1+4n\right)} \right\}_{n=1}^{\infty}$

Section 4-2 : More on Sequences

For each of the following problems determine if the sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

1. $\left\{\dfrac{1}{4n}\right\}_{n=1}^{\infty}$

2. $\left\{n\left(-1\right)^{n+2}\right\}_{n=0}^{\infty}$

3. $\left\{3^{-n}\right\}_{n=0}^{\infty}$

4. $\left\{\dfrac{2n^2-1}{n}\right\}_{n=2}^{\infty}$

5. $\left\{\dfrac{4-n}{2n+3}\right\}_{n=1}^{\infty}$

6. $\left\{\dfrac{-n}{n^2+25}\right\}_{n=2}^{\infty}$

Section 4-3 : Series - The Basics

For problems 1 – 3 perform an index shift so that the series starts at $n = 3$.

1. $\displaystyle\sum_{n=1}^{\infty}\left(n2^{n} - 3^{1-n}\right)$

2. $\displaystyle\sum_{n=7}^{\infty}\frac{4-n}{n^2+1}$

3. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n-3}(n+2)}{5^{1+2n}}$

4. Strip out the first 3 terms from the series $\displaystyle\sum_{n=1}^{\infty}\frac{2^{-n}}{n^2+1}$.

5. Given that $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n^3+1} = 1.6865$ determine the value of $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^3+1}$.

Section 4-4 : Convergence/Divergence of Series

For problems 1 & 2 compute the first 3 terms in the sequence of partial sums for the given series.

1. $\displaystyle\sum_{n=1}^{\infty} n\,2^n$

2. $\displaystyle\sum_{n=3}^{\infty} \frac{2n}{n+2}$

For problems 3 & 4 assume that the n^{th} term in the sequence of partial sums for the series $\displaystyle\sum_{n=0}^{\infty} a_n$ is given below. Determine if the series $\displaystyle\sum_{n=0}^{\infty} a_n$ is convergent or divergent. If the series is convergent determine the value of the series.

3. $s_n = \dfrac{5+8n^2}{2-7n^2}$

4. $s_n = \dfrac{n^2}{5+2n}$

For problems 5 & 6 show that the series is divergent.

5. $\displaystyle\sum_{n=0}^{\infty} \frac{3n\,\mathbf{e}^n}{n^2+1}$

6. $\displaystyle\sum_{n=5}^{\infty} \frac{6+8n+9n^2}{3+2n+n^2}$

Section 4-5 : Special Series

For each of the following series determine if the series converges or diverges. If the series converges give its value.

1. $\displaystyle\sum_{n=0}^{\infty} 3^{2+n}\,2^{1-3n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{6n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{\left(-6\right)^{3-n}}{8^{2-n}}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2+7n+12}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{5^{n+1}}{7^{n-2}}$

6. $\displaystyle\sum_{n=2}^{\infty} \frac{5^{n+1}}{7^{n-2}}$

7. $\displaystyle\sum_{n=4}^{\infty} \frac{10}{n^2-4n+3}$

Section 4-6 : Integral Test

For each of the following series determine if the series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{2}{3+5n}$

3. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\left(2n+7\right)^3}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{n^3+1}$

5. $\displaystyle\sum_{n=3}^{\infty} \frac{3}{n^2-3n+2}$

Section 4-7 : Comparison Test/Limit Comparison Test

For each of the following series determine if the series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+1\right)^2$

2. $\displaystyle\sum_{n=4}^{\infty}\frac{n^2}{n^3-3}$

3. $\displaystyle\sum_{n=2}^{\infty}\frac{7}{n(n+1)}$

4. $\displaystyle\sum_{n=7}^{\infty}\frac{4}{n^2-2n-3}$

5. $\displaystyle\sum_{n=2}^{\infty}\frac{n-1}{\sqrt{n^6+1}}$

6. $\displaystyle\sum_{n=1}^{\infty}\frac{2n^3+7}{n^4\sin^2(n)}$

7. $\displaystyle\sum_{n=0}^{\infty}\frac{2^n\sin^2(5n)}{4^n+\cos^2(n)}$

8. $\displaystyle\sum_{n=3}^{\infty}\frac{\mathbf{e}^{-n}}{n^2+2n}$

9. $\displaystyle\sum_{n=1}^{\infty}\frac{4n^2-n}{n^3+9}$

10. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{2n^2+4n+1}}{n^3+9}$

Section 4-8 : Alternating Series Test

For each of the following series determine if the series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{7+2n}$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+3}}{n^3+4n+1}$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(-1)^n\left(2^n+3^n\right)}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+6}\,n}{n^2+9}$

5. $\displaystyle\sum_{n=4}^{\infty} \frac{(-1)^{n+2}(1-n)}{3n-n^2}$

Section 4-9 : Absolute Convergence

For each of the following series determine if they are absolutely convergent, conditionally convergent or divergent.

1. $\displaystyle\sum_{n=2}^{\infty} \frac{\left(-1\right)^{n+1}}{n^3+1}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{\left(-1\right)^{n-3}}{\sqrt{n}}$

3. $\displaystyle\sum_{n=3}^{\infty} \frac{\left(-1\right)^{n+1}\left(n+1\right)}{n^3+1}$

Section 4-10 : Ratio Test

For each of the following series determine if the series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{3^{1-2n}}{n^2+1}$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{5n+1}$

3. $\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^{1+3n}(n+1)}{n^2 5^{1+n}}$

4. $\displaystyle\sum_{n=3}^{\infty} \frac{\mathbf{e}^{4n}}{(n-2)!}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{6n+7}$

Section 4-11 : Root Test

For each of the following series determine if the series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3n+1}{4-2n}\right)^{2n}$

2. $\displaystyle\sum_{n=0}^{\infty}\frac{n^{1-3n}}{4^{2n}}$

3. $\displaystyle\sum_{n=4}^{\infty}\frac{\left(-5\right)^{1+2n}}{2^{5n-3}}$

Section 4-12 : Strategy for Series

Problems have not yet been written for this section.

I was finding it very difficult to come up with a good mix of "new" problems and decided my time was better spent writing problems for later sections rather than trying to come up with a sufficient number of problems for what is essentially a review section. I intend to come back at a later date when I have more time to devote to this section and add problems then.

Section 4-13 : Estimating the Value of a Series

1. Use the Integral Test and $n = 10$ to estimate the value of $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\left(n^2+1\right)^2}$.

2. Use the Comparison Test and $n = 20$ to estimate the value of $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^3 \ln\left(n\right)}$.

3. Use the Alternating Series Test and $n = 16$ to estimate the value of $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2+1}$.

4. Use the Ratio Test and $n = 8$ to estimate the value of $\displaystyle\sum_{n=1}^{\infty} \frac{3^{1+n}}{n\, 2^{3+2n}}$.

Section 4-14 : Power Series

For each of the following power series determine the interval and radius of convergence.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(-3)^{2+n}(n^2+1)}(4x-12)^n$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{n^{2n+1}}{4^{3n}}(2x+17)^n$

3. $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{(2n+1)!}(x-2)^n$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{4^{1+2n}}{5^{n+1}}(x+3)^n$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{6^n}{n}(4x-1)^{n-1}$

Section 4-15 : Power Series and Functions

For problems 1 – 3 write the given function as a power series and give the interval of convergence.

1. $f\left(x\right)=\dfrac{6}{1+7x^4}$

2. $f\left(x\right)=\dfrac{x^3}{3-x^2}$

3. $f\left(x\right)=\dfrac{3x^2}{5-2\sqrt[3]{x}}$

4. Give a power series representation for the derivative of the following function.

$$g\left(x\right)=\frac{5x}{1-3x^5}$$

5. Give a power series representation for the integral of the following function.

$$h\left(x\right)=\frac{x^4}{9+x^2}$$

Section 4-16 : Taylor Series

For problems 1 & 2 use one of the Taylor Series derived in the notes to determine the Taylor Series for the given function.

1. $f\left(x\right)=\cos\left(4x\right)$ about $x=0$

2. $f\left(x\right)=x^6\mathbf{e}^{2x^3}$ about $x=0$

For problem 3 – 6 find the Taylor Series for each of the following functions.

3. $f\left(x\right)=\mathbf{e}^{-6x}$ about $x=-4$

4. $f\left(x\right)=\ln\left(3+4x\right)$ about $x=0$

5. $f\left(x\right)=\dfrac{7}{x^4}$ about $x=-3$

6. $f\left(x\right)=7x^2-6x+1$ about $x=2$

Section 4-17 : Applications of Series

1. Determine a Taylor Series about $x = 0$ for the following integral.

$$\int \frac{\mathbf{e}^{x} - 1}{x}\,dx$$

2. Write down $T_2(x)$, $T_3(x)$ and $T_4(x)$ for the Taylor Series of $f(x) = \mathbf{e}^{-6x}$ about $x = -4$. Graph all three of the Taylor polynomials and $f(x)$ on the same graph for the interval $[-8, -2]$.

3. Write down $T_3(x)$, $T_4(x)$ and $T_5(x)$ for the Taylor Series of $f(x) = \ln(3 + 4x)$ about $x = 0$. Graph all three of the Taylor polynomials and $f(x)$ on the same graph for the interval $\left[-\frac{1}{2}, 2\right]$.

Section 4-18 : Binomial Series

For problems 1 & 2 use the Binomial Theorem to expand the given function.

1. $\left(4+3x\right)^{5}$

2. $\left(9-x\right)^{4}$

For problems 3 and 4 write down the first four terms in the binomial series for the given function.

3. $\left(1+3x\right)^{-6}$

4. $\sqrt[3]{8-2x}$