Section 4-10 : Ratio Test

1. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3^{1-2n}}{n^2+1}$$

Step 1
We'll need to compute *L*.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{1-2(n+1)}}{(n+1)^2+1}\frac{n^2+1}{3^{1-2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{3^{-1-2n}}{(n+1)^2+1}\frac{n^2+1}{3^{1-2n}}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)^2+1}\frac{n^2+1}{3^2}\right| = \lim_{n\to\infty}\left|\frac{n^2+1}{9\left[(n+1)^2+1\right]}\right| = \frac{1}{9}$$

When computing a_{n+1} be careful to pay attention to any coefficients of *n* and powers of *n*. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2
Okay, we can see that $L = \frac{1}{9} < 1$ and so by the Ratio Test the series **converges**.

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2. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{(2n)!}{5n+1}$$

Step 1
We'll need to compute *L*.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(2(n+1)\right)!}{5(n+1)+1}\frac{5n+1}{(2n)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2n+2)!}{5n+6}\frac{5n+1}{(2n)!}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)(2n)!}{5n+6}\frac{5n+1}{(2n)!}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)(5n+1)}{5n+6}\right| = \infty$$

When computing a_{n+1} be careful to pay attention to any coefficients of *n* and powers of *n*. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2
Okay, we can see that $L = \infty > 1$ and so by the Ratio Test the series **diverges**.

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3. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{(-2)^{1+3n}(n+1)}{n^2 5^{1+n}}$$

Step 1
We'll need to compute *L*.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{1+3(n+1)}(n+1+1)}{(n+1)^2 5^{1+n+1}}\frac{n^2 5^{1+n}}{(-2)^{1+3n}(n+1)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-2)^{4+3n}(n+2)}{(n+1)^2 5^{2+n}}\frac{n^2 5^{1+n}}{(-2)^{1+3n}(n+1)}\right| = \lim_{n\to\infty}\left|\frac{(-2)^3(n+2)}{(n+1)^2(5)}\frac{n^2}{(n+1)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{-8n^2(n+2)}{5(n+1)^2(n+1)}\right| = \frac{8}{5}$$

When computing a_{n+1} be careful to pay attention to any coefficients of *n* and powers of *n*. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2
Okay, we can see that $L = \frac{8}{5} > 1$ and so by the Ratio Test the series **diverges**.

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4. Determine if the following series converges or diverges.

$$\sum_{n=3}^{\infty} \frac{\mathbf{e}^{4n}}{(n-2)!}$$

Step 1

We'll need to compute *L*.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\mathbf{e}^{4(n+1)}}{(n+1-2)!}\frac{(n-2)!}{\mathbf{e}^{4n}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{\mathbf{e}^{4n+4}}{(n-1)!}\frac{(n-2)!}{\mathbf{e}^{4n}}\right| = \lim_{n\to\infty}\left|\frac{\mathbf{e}^{4n+4}}{(n-1)(n-2)!}\frac{(n-2)!}{\mathbf{e}^{4n}}\right| = \lim_{n\to\infty}\left|\frac{\mathbf{e}^{4}}{n-1}\right| = 0$$

When computing a_{n+1} be careful to pay attention to any coefficients of *n* and powers of *n*. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2
Okay, we can see that $L = 0 < 1$ and so by the Ratio Test the series **converges**.

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5. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{6n+7}$$

Step 1
We'll need to compute *L*.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|a_{n+1}\frac{1}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1+1}}{6(n+1)+7}\frac{6n+7}{(-1)^{n+1}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(-1)^{n+2}}{6n+13}\frac{6n+7}{(-1)^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)(6n+7)}{6n+13}\right| = 1$$

When computing a_{n+1} be careful to pay attention to any coefficients of *n* and powers of *n*. Failure to properly deal with these is one of the biggest mistakes that students make in this computation and mistakes at that level often lead to the wrong answer!

Step 2
Okay, we can see that $L = 1$ and so by the Ratio Test tells us nothing about this series.

Step 3
Just because the Ratio Test doesn't tell us anything doesn't mean we can't determine if this series converges or diverges.

In fact, it's actually quite simple to do in this case. This is an Alternating Series with,

$$b_n = \frac{1}{6n+7}$$

The b_n are clearly positive and it should be pretty obvious (hopefully) that they also form a decreasing sequence. Finally, we also can see that $\lim_{n\to\infty} b_n = 0$ and so by the Alternating Series Test this series will **converge**.

Note, that if this series were not in this section doing this as an Alternating Series from the start would probably have been the best way of approaching this problem.
