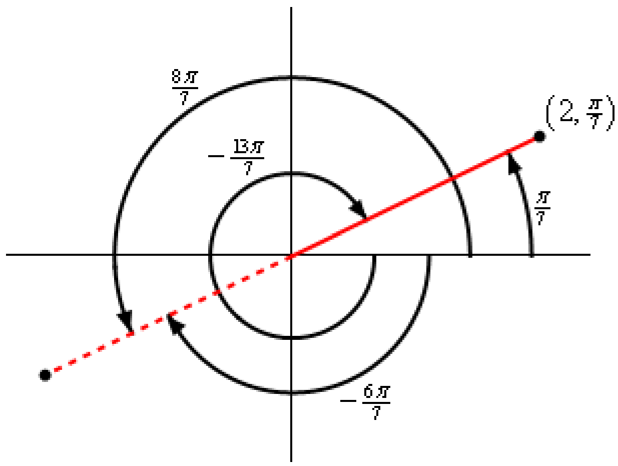
Section 3-6 : Polar Coordinates

1. For the point with polar coordinates $\left(2,\frac{\pi}{7}\right)$ determine three different sets of coordinates for the same point all of which have angles different from $\frac{\pi}{7}$ and are in the range $-2\pi \le \theta \le 2\pi$.

Step 1
This problem is really as exercise in how well we understand the unit circle. Here is a quick sketch of the point and some angles.



We can see that the negative angle ending at the solid red line that is in the range specified in the problem statement is simply $\frac{\pi}{7}-2\pi=-\frac{13\pi}{7}$.

If we extend the solid line into the third quadrant (*i.e.* the dashed red line) then the positive angle ending at the dashed red line is $\frac{\pi}{7}+\pi=\frac{8\pi}{7}$. Likewise, the negative angle ending at the dashed red line is $\frac{\pi}{7}-\pi=-\frac{6\pi}{7}$.

With these angles getting the other three points should be pretty simple.

Step 2
For the first "new" point we can use the negative angle that ends on the solid red line to get the point.

$$\boxed{\left(2,-\frac{13\pi}{7}\right)}$$

Step 3
For the remaining points recall that if we use a negative *r* then we go "backwards" from where the angle ends to get the point. So, if we use $r=-2$, any angle that ends on the dashed red line will go "backwards" into the first quadrant 2 units to get to the point.

This gives the remaining two points using both the positive and negative angle ending on the dashed red line,

$\boxed{\left(-2,-\frac{6\pi}{7}\right)}$ $\boxed{\left(-2,\frac{8\pi}{7}\right)}$

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2. The polar coordinates of a point are $\left(-5,0.23\right)$. Determine the Cartesian coordinates for the point.

Solution
There really isn't too much to this problem. From the point we can see that we have $r=-5$ and $\theta=0.23$ (in radians of course!). Once we have these all we need to is plug into the formulas from this section to get,

$$x=r\cos\theta=\left(-5\right)\cos\left(0.23\right)=-4.8683 \qquad y=r\sin\theta=\left(-5\right)\sin\left(0.23\right)=-1.1399$$

So, the Cartesian coordinates for the point are then,

$$\boxed{\left(-4.8683,-1.1399\right)}$$

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3. The Cartesian coordinate of a point are $\left(2,-6\right)$. Determine a set of polar coordinates for the point.

Step 1
Let's first determine *r*. That's always simple.

$$r=\sqrt{x^2+y^2}=\sqrt{\left(2\right)^2+\left(-6\right)^2}=\sqrt{40}=2\sqrt{10}$$

Step 2
Next let's get θ. As we do this we need to remember that we actually have two possible values of which only one will work with the *r* we found in the first step.

Here are the two possible values of θ.

$$\theta_1=\tan^{-1}\left(\frac{y}{x}\right)=\tan^{-1}\left(\frac{-6}{2}\right)=-1.2490 \qquad \theta_2=\theta_1+\pi=1.8926$$

So, we can see that $-\frac{\pi}{2}=-1.57<\theta_1=-1.2490<0$ and so θ_1 is in the fourth quadrant. Likewise, $\frac{\pi}{2}=1.57<\theta_2=1.8926<\pi=3.14$ and so θ_2 is in the second quadrant.

We can also see from the Cartesian coordinates of the point that our point must be in the fourth quadrant and so, for this problem, θ_1 is the correct value.

The polar coordinates of the point using the *r* from the first step and θ from this step is,

$$\boxed{\left(2\sqrt{10},-1.2490\right)}$$

Note of course that there are many other sets of polar coordinates that are just as valid for this point. These are simply the set that we get from the formulas discussed in this section.

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4. The Cartesian coordinate of a point are $\left(-8,1\right)$. Determine a set of polar coordinates for the point.

Step 1
Let's first determine *r*. That's always simple.

$$r=\sqrt{x^2+y^2}=\sqrt{\left(-8\right)^2+\left(1\right)^2}=\sqrt{65}$$

Step 2
Next let's get θ. As we do this we need to remember that we actually have two possible values of which only one will work with the *r* we found in the first step.

Here are the two possible values of θ.

$$\theta_1=\tan^{-1}\left(\tfrac{y}{x}\right)=\tan^{-1}\left(\tfrac{1}{-8}\right)=-0.1244 \qquad\qquad \theta_2=\theta_1+\pi=3.0172$$

So, we can see that $-\frac{\pi}{2}=-1.57<\theta_1=-0.1244<0$ and so θ_1 is in the fourth quadrant. Likewise, $\frac{\pi}{2}=1.57<\theta_2=3.0172<\pi=3.14$ and so θ_2 is in the second quadrant.

We can also see from the Cartesian coordinates of the point that our point must be in the second quadrant and so, for this problem, θ_2 is the correct value.

The polar coordinates of the point using the *r* from the first step and θ from this step is,

$$\boxed{\left(\sqrt{65},\ 3.0172\right)}$$

Note of course that there are many other sets of polar coordinates that are just as valid for this point. These are simply the set that we get from the formulas discussed in this section.

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5. Convert the following equation into an equation in terms of polar coordinates.

$$\frac{4x}{3x^2+3y^2}=6-xy$$

Solution
Basically, what we need to do here is to convert all the *x*'s and *y*'s into *r*'s and θ's using the following formulas.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2$$

Don't forget about the last one! If it is possible to use this formula (and you can see where we'll use it in the case can't you?) it will save a lot of work!

First let's substitute in the equations as needed.

$$\frac{4\left(r\cos\theta\right)}{3r^2}=6-\left(r\cos\theta\right)\left(r\sin\theta\right)$$

Finally, as we need to do is take care of little simplification to get,

$$\boxed{\frac{4\cos\theta}{3r}=6-r^2\cos\theta\sin\theta}$$

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6. Convert the following equation into an equation in terms of polar coordinates.

$$x^2=\frac{4x}{y}-3y^2+2$$

Solution
Basically, what we need to do here is to convert all the *x*'s and *y*'s into *r*'s and θ's using the following formulas.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2$$

Don't forget about the last one! If it is possible to use this formula (which won't do us a lot of good in this problem) it will save a lot of work!

First let's substitute in the equations as needed.

$$\left(r\cos\theta\right)^2=\frac{4\left(r\cos\theta\right)}{r\sin\theta}-3\left(r\sin\theta\right)^2+2$$

Finally, as we need to do is take care of little simplification to get,

$$\boxed{r^2\cos^2\theta = 4\cot\theta - 3r^2\sin^2\theta + 2}$$

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7. Convert the following equation into an equation in terms of Cartesian coordinates.

$$6r^3\sin\theta = 4 - \cos\theta$$

Solution
There is a variety of ways to work this problem. One way is to first multiply everything by *r* and then doing a little rearranging as follows,

$$6r^4\sin\theta = 4r - r\cos\theta \qquad \Rightarrow \qquad 6r^3\left(r\sin\theta\right) = 4r - r\cos\theta$$

We can now use the following formulas to finish this problem.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2 + y^2}$$

Here is the answer for this problem,

$$\boxed{6y\left[\sqrt{x^2 + y^2}\,\right]^3 = 4\sqrt{x^2 + y^2} - x}$$

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8. Convert the following equation into an equation in terms of Cartesian coordinates.

$$\frac{2}{r} = \sin\theta - \sec\theta$$

Solution
There is a variety of ways to work this problem. One way is to first do the following rearranging/rewriting of the equation.

$$\frac{2}{r} = \sin\theta - \frac{1}{\cos\theta} \qquad \to \qquad \frac{2\cos\theta}{r} = \sin\theta\cos\theta - 1$$

At this point we can multiply everything by r^2 and do a little rearranging as follows,

$$2r\cos\theta = r^2\sin\theta\cos\theta - r^2 \qquad \to \qquad 2r\cos\theta = \left(r\sin\theta\right)\left(r\cos\theta\right) - r^2$$

We can now use the following formulas to finish this problem.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2$$

Here is the answer for this problem,

$$\boxed{2x = yx - \left(x^2 + y^2\right)}$$

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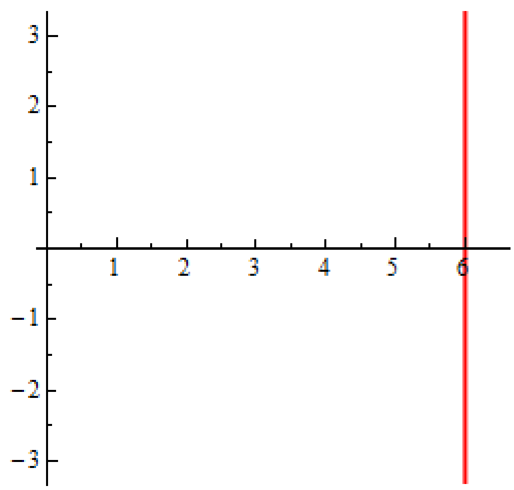
9. Sketch the graph of the following polar equation.

$$\cos\theta = \frac{6}{r}$$

Solution
Multiplying both sides by *r* gives,

$$r\cos\theta = 6$$

and we know from the notes on this section that this is simply the vertical line $x = 6$. So here is the graph of this function.



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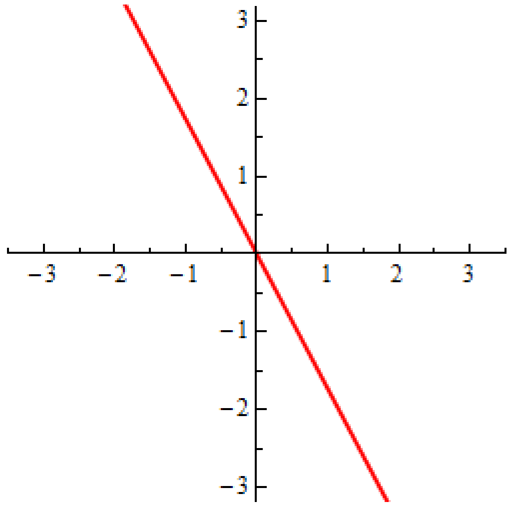
10. Sketch the graph of the following polar equation.

$$\theta = -\frac{\pi}{3}$$

Solution
We know from the notes on this section that this is simply the line that goes through the origin and has slope of $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$.

So here is the graph of this function.



11. Sketch the graph of the following polar equation.

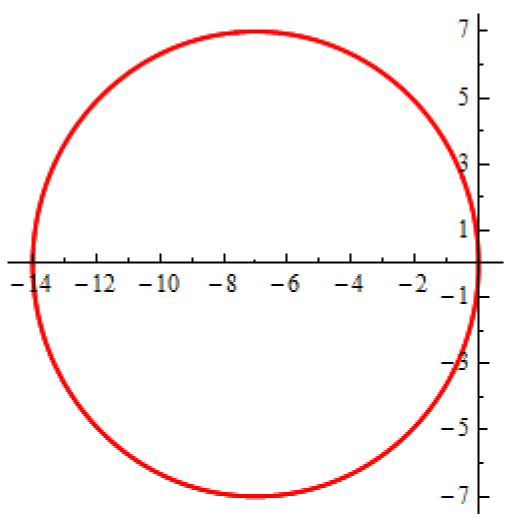
$$r = -14\cos\theta$$

Solution
We can rewrite this as,

$$r = 2\left(-7\right)\cos\theta$$

and so we know from the notes on this section that this is simply the circle with radius 7 and center $\left(-7,0\right)$.

So here is the graph of this function.

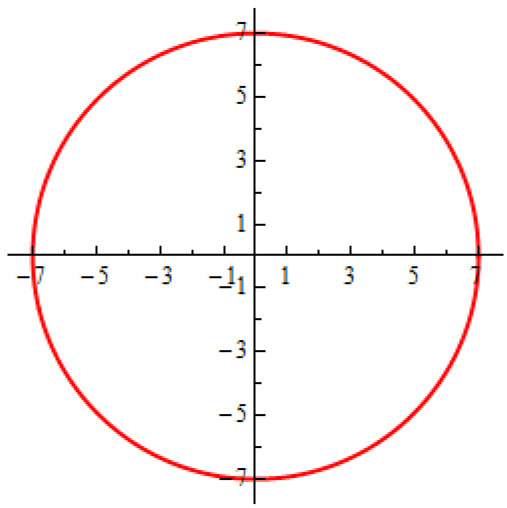


12. Sketch the graph of the following polar equation.

$$r = 7$$

Solution
We know from the notes on this section that this is simply the circle with radius 7 and centered at the origin.

So here is the graph of this function.



13. Sketch the graph of the following polar equation.

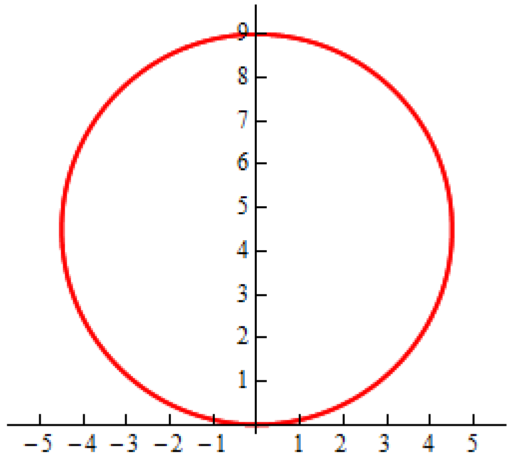
$$r = 9\sin\theta$$

Solution
We can rewrite this as,

$$r = 2\left(\tfrac{9}{2}\right)\sin\theta$$

and so we know from the notes on this section that this is simply the circle with radius $\frac{9}{2}$ and center $\left(0, \frac{9}{2}\right)$.

So here is the graph of this function.



14. Sketch the graph of the following polar equation.

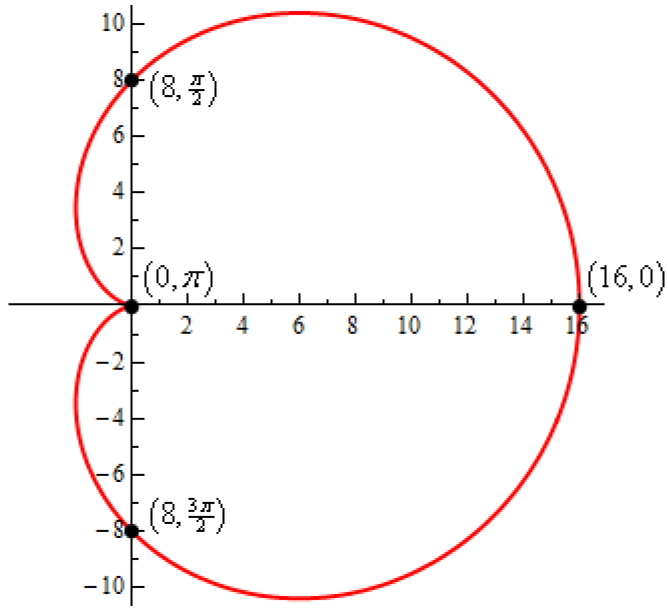
$$r = 8 + 8\cos\theta$$

Solution

We know from the notes on this section that this is a cardioid and so all we really need to get the graph is a quick chart of points.

θ	r	(r,θ)
0	16	$(16,0)$
$\frac{\pi}{2}$	8	$\left(8,\frac{\pi}{2}\right)$
π	0	$(0,\pi)$
$\frac{3\pi}{2}$	8	$\left(8,\frac{3\pi}{2}\right)$
2π	16	$(16,2\pi)$

So here is the graph of this function.



Be careful when plotting these points and remember the rules for graphing polar coordinates. The "tick marks" on the graph are really the Cartesian coordinate tick marks because those are the ones we are familiar with. Do not let them confuse you when you go to plot the polar points for our sketch.

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15. Sketch the graph of the following polar equation.

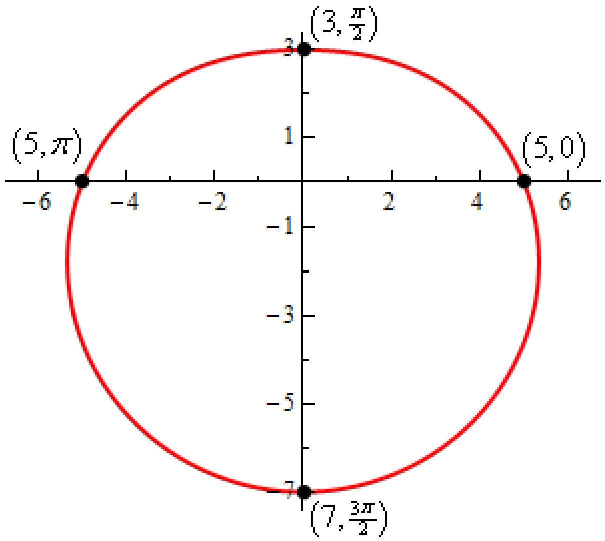
$$r = 5 - 2\sin\theta$$

Solution

We know from the notes on this section that this is a limacon without an inner loop and so all we really need to get the graph is a quick chart of points.

θ	r	(r,θ)
0	5	$(5,0)$
$\frac{\pi}{2}$	3	$(3,\frac{\pi}{2})$
π	5	$(5,\pi)$
$\frac{3\pi}{2}$	7	$(7,\frac{3\pi}{2})$
2π	5	$(5,2\pi)$

So here is the graph of this function.



Be careful when plotting these points and remember the rules for graphing polar coordinates. The "tick marks" on the graph are really the Cartesian coordinate tick marks because those are the ones we are familiar with. Do not let them confuse you when you go to plot the polar points for our sketch.

Also, many of these graphs are vaguely heart shaped although as this sketch has shown many do and this one is more circular than heart shaped.

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16. Sketch the graph of the following polar equation.

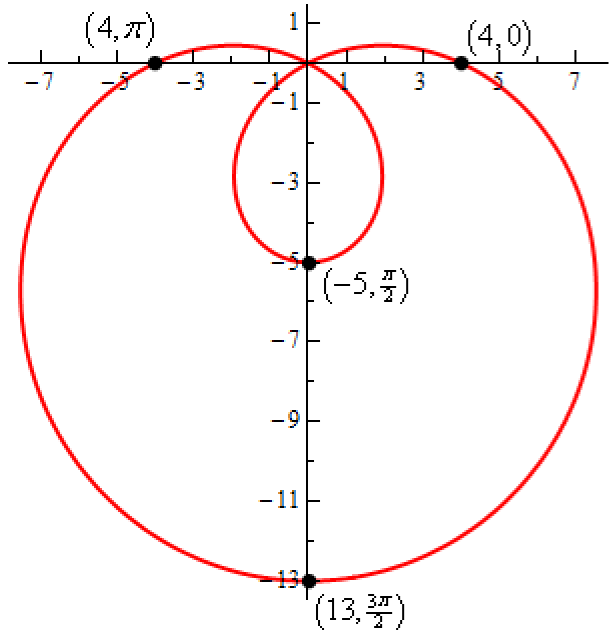
$$r = 4 - 9\sin\theta$$

Solution

We know from the notes on this section that this is a limacon with an inner loop and so all we really need to get the graph is a quick chart of points.

θ	r	(r,θ)
0	4	$(4,0)$
$\frac{\pi}{2}$	-5	$\left(-5,\frac{\pi}{2}\right)$
π	4	$(4,\pi)$
$\frac{3\pi}{2}$	13	$\left(13,\frac{3\pi}{2}\right)$
2π	4	$(4,2\pi)$

So here is the graph of this function.



Be careful when plotting these points and remember the rules for graphing polar coordinates. The "tick marks" on the graph are really the Cartesian coordinate tick marks because those are the ones we are familiar with. Do not let them confuse you when you go to plot the polar points for our sketch.

Section 3-7 : Tangents with Polar Coordinates

1. Find the tangent line to $r = \sin(4\theta)\cos(\theta)$ at $\theta = \frac{\pi}{6}$.

Step 1

First, we'll need to following derivative,

$$\frac{dr}{d\theta} = 4\cos(4\theta)\cos(\theta) - \sin(4\theta)\sin(\theta)$$

Step 2

Next using the formula from the notes on this section we have,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\ &= \frac{\left(4\cos(4\theta)\cos(\theta) - \sin(4\theta)\sin(\theta)\right)\sin\theta + \left(\sin(4\theta)\cos(\theta)\right)\cos\theta}{\left(4\cos(4\theta)\cos(\theta) - \sin(4\theta)\sin(\theta)\right)\cos\theta - \left(\sin(4\theta)\cos(\theta)\right)\sin\theta}\end{aligned}$$

This is a very messy derivative (these often are) and, at least in this case, there isn't a lot of simplification that we can do…

Step 3

Next, we'll need to evaluate both the derivative from the previous step as well as *r* at $\theta = \frac{\pi}{6}$.

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{1}{3\sqrt{3}} \qquad\qquad r\big|_{\theta=\frac{\pi}{6}} = \frac{3}{4}$$

You can see why we need both of these right?

Step 4

Last, we need the *x* and *y* coordinate that we'll be at when $\theta = \frac{\pi}{6}$. These values are easy enough to find given that we know what *r* is at this point and we also know the polar to Cartesian coordinate conversion formulas. So,

$$x = r\cos(\theta) = \tfrac{3}{4}\cos\left(\tfrac{\pi}{6}\right) = \tfrac{3\sqrt{3}}{8} \qquad\qquad y = r\sin(\theta) = \tfrac{3}{4}\sin\left(\tfrac{\pi}{6}\right) = \tfrac{3}{8}$$

Of course, we also have the slope of the tangent line since it is just the value of the derivative we computed in the previous step.

Step 5
The tangent line is then,

$$\boxed{y = \tfrac{3}{8} + \tfrac{1}{3\sqrt{3}}\left(x - \tfrac{3\sqrt{3}}{8}\right) = \tfrac{1}{3\sqrt{3}}x + \tfrac{1}{4}}$$

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2. Find the tangent line to $r = \theta - \cos\left(\theta\right)$ at $\theta = \dfrac{3\pi}{4}$.

Step 1
First, we'll need to following derivative,

$$\frac{dr}{d\theta} = 1 + \sin\left(\theta\right)$$

Step 2
Next using the formula from the notes on this section we have,

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\ &= \frac{\left(1+\sin\left(\theta\right)\right)\sin\theta + \left(\theta - \cos\left(\theta\right)\right)\cos\theta}{\left(1+\sin\left(\theta\right)\right)\cos\theta - \left(\theta - \cos\left(\theta\right)\right)\sin\theta} \end{aligned}$$

This is a somewhat messy derivative (these often are) and, at least in this case, there isn't a lot of simplification that we can do…

Step 3
Next, we'll need to evaluate both the derivative from the previous step as well as *r* at $\theta = \dfrac{3\pi}{4}$.

$$\left.\frac{dy}{dx}\right|_{\theta = \frac{3\pi}{4}} = 0.2843 \qquad\qquad \left.r\right|_{\theta = \frac{3\pi}{4}} = 3.0633$$

You can see why we need both of these right?

Step 4

Last, we need the *x* and *y* coordinate that we'll be at when $\theta = \dfrac{3\pi}{4}$. These values are easy enough to find given that we know what *r* is at this point and we also know the polar to Cartesian coordinate conversion formulas. So,

$$x = r\cos\left(\theta\right) = 3.0633\cos\left(\tfrac{3\pi}{4}\right) = -2.1661 \qquad y = r\sin\left(\theta\right) = 3.0633\sin\left(\tfrac{3\pi}{4}\right) = 2.1661$$

Of course, we also have the slope of the tangent line since it is just the value of the derivative we computed in the previous step.

Step 5

The tangent line is then,

$$\boxed{y = 2.1661 + 0.2843\left(x + 2.1661\right) = 0.2843x + 2.7819}$$

