Section 3-4 : Arc Length with Parametric Equations

1. Determine the length of the parametric curve given by the following set of parametric equations. You may assume that the curve traces out exactly once for the given range of t's.

$$x = 8t^{\frac{3}{2}} \qquad y = 3 + (8-t)^{\frac{3}{2}} \qquad 0 \le t \le 4$$

Step 1

The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 12t^{\frac{1}{2}} \qquad\qquad \frac{dy}{dt} = -\tfrac{3}{2}(8-t)^{\frac{1}{2}}$$

Step 2

We'll need the *ds* for this problem.

$$ds = \sqrt{\left[12t^{\frac{1}{2}}\right]^2 + \left[-\tfrac{3}{2}(8-t)^{\frac{1}{2}}\right]^2}\,dt = \sqrt{144t + \tfrac{9}{4}(8-t)}\,dt = \sqrt{\tfrac{567}{4}t + 18}\,dt$$

Step 3

The integral for the arc length is then,

$$L = \int ds = \int_0^4 \sqrt{\tfrac{567}{4}t + 18}\,dt$$

Step 4

This is a simple integral to compute with a quick substitution. Here is the integral work,

$$L = \int_0^4 \sqrt{\tfrac{567}{4}t + 18}\,dt = \tfrac{4}{567}\left(\tfrac{2}{3}\right)\left(\tfrac{567}{4}t + 18\right)^{\frac{3}{2}}\Big|_0^4 = \boxed{\tfrac{8}{1701}\left(585^{\frac{3}{2}} - 18^{\frac{3}{2}}\right) = 66.1865}$$

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2. Determine the length of the parametric curve given by the following set of parametric equations. You may assume that the curve traces out exactly once for the given range of t's.

$$x = 3t + 1 \qquad y = 4 - t^2 \qquad -2 \le t \le 0$$

Step 1

The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 3 \qquad\qquad \frac{dy}{dt} = -2t$$

Step 2

We'll need the *ds* for this problem.

$$ds = \sqrt{[3]^2 + [-2t]^2}\,dt = \sqrt{9+4t^2}\,dt$$

Step 3
The integral for the arc length is then,

$$L = \int ds = \int_{-2}^{0} \sqrt{9+4t^2}\,dt$$

Step 4
This integral will require a trig substitution (as will quite a few arc length integrals!).

Here is the trig substitution we'll need for this integral.

$$t = \tfrac{3}{2}\tan\theta \qquad dt = \tfrac{3}{2}\sec^2\theta\,d\theta$$
$$\sqrt{9+4t^2} = \sqrt{9+9\tan^2\theta} = 3\sqrt{1+\tan^2\theta} = 3\sqrt{\sec^2\theta} = 3\left|\sec\theta\right|$$

To get rid of the absolute value on the secant will need to convert the limits into θ limits.

$$t=-2: \qquad -2 = \tfrac{3}{2}\tan\theta \quad\rightarrow\quad \tan\theta = -\tfrac{4}{3} \quad\rightarrow\quad \theta = \tan^{-1}\left(-\tfrac{4}{3}\right) = -0.9273$$
$$t=0: \qquad 0 = \tfrac{3}{2}\tan\theta \quad\rightarrow\quad \tan\theta = 0 \quad\rightarrow\quad \theta = 0$$

Okay, the corresponding range of θ for this problem is $-0.9273 \le \theta \le 0$ (fourth quadrant) and in this range we know that secant is positive. Therefore the root becomes,

$$\sqrt{9+4t^2} = 3\sec\theta$$

The integral is then,

$$\begin{aligned} L &= \int_{-2}^{0} \sqrt{9+4t^2}\,dt = \int_{-0.9273}^{0} \left(3\sec\theta\right)\left(\tfrac{3}{2}\sec^2\theta\right)d\theta \\ &= \int_{-0.9273}^{0} \tfrac{9}{2}\sec^3\theta\,d\theta = \tfrac{9}{4}\Big[\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\Big]\Big|_{-0.9273}^{0} = \boxed{7.4719} \end{aligned}$$

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3. A particle travels along a path defined by the following set of parametric equations. Determine the total distance the particle travels and compare this to the length of the parametric curve itself.

$$x = 4\sin\left(\tfrac{1}{4}t\right) \qquad y = 1 - 2\cos^2\left(\tfrac{1}{4}t\right) \qquad\qquad -52\pi \le t \le 34\pi$$

Hint : Be very careful with this problem. Note the two quantities we are being asked to find, how they relate to each other and which of the two that we know how to compute from the material in this section.

Step 1

This is a problem that many students have issues with. First note that we are being asked to find both the total distance traveled by the particle AND the length of the curve. Also, recall that of these two quantities we only discussed how to determine the length of a curve in this section.

Therefore, let's concentrate on finding the length of the curve first, then we'll worry about the total distance traveled.

Step 2

To find the length we'll need the following two derivatives,

$$\frac{dx}{dt} = \cos\left(\tfrac{1}{4}t\right) \qquad\qquad \frac{dy}{dt} = \cos\left(\tfrac{1}{4}t\right)\sin\left(\tfrac{1}{4}t\right)$$

The *ds* for this problem is then,

$$ds = \sqrt{\left[\cos\left(\tfrac{1}{4}t\right)\right]^2 + \left[\cos\left(\tfrac{1}{4}t\right)\sin\left(\tfrac{1}{4}t\right)\right]^2}\,dt = \sqrt{\cos^2\left(\tfrac{1}{4}t\right) + \cos^2\left(\tfrac{1}{4}t\right)\sin^2\left(\tfrac{1}{4}t\right)}\,dt$$

Now, this is where many students run into issues with this problem. Many students use the following integral to determine the length of the curve.

$$\int_{-52\pi}^{34\pi} \sqrt{\cos^2\left(\tfrac{1}{4}t\right) + \cos^2\left(\tfrac{1}{4}t\right)\sin^2\left(\tfrac{1}{4}t\right)}\,dt$$

Can you see what is wrong with this integral?

Step 3

Remember from the discussion in this section that in order to use the arc length formula the curve can only trace out exactly once over the range of the limits in the integral.

Therefore, we can't even write down the formula that we did in the previous step until we first determine if the curve traces out exactly once in the given range of *t*'s.

If it turns out that the curve traces out more than once in the given range of *t*'s then the integral we wrote down in the previous step is simply wrong. We will then need to determine a range of *t*'s for one trace so we can write down the proper integral for the length.

Luckily enough for us we actually did this in a practice problem in the Parametric Equations and Curves section. Examining this set of parametric equations was problem #8 from that section. From the solution to that problem we found a couple of pieces of information that will be needed for this problem.

First, we determined that the curve does trace out more than once in the given range of *t*'s for the problem. In fact, we determined that the curve traced out 21.5 times over the given range of *t*'s.

Secondly, and more importantly at this point, we also determined that the curve would trace out exactly once in the range of $-2\pi \le t \le 2\pi$. Note that we actually listed several possible ranges of *t*'s for one trace and we can use any of them. This is simply the first one found and so it's the one we decided to use for this problem.

So, we can now see that the integral we wrote down in the previous step was in fact not correct and will not give the length of the curve.

Using the range of *t*'s we found in the earlier problem we can see that the integral for the length is,

$$L = \int_{-2\pi}^{2\pi} \sqrt{\cos^2\left(\tfrac{1}{4}t\right) + \cos^2\left(\tfrac{1}{4}t\right)\sin^2\left(\tfrac{1}{4}t\right)}\,dt$$

Before proceeding with the next step we should address the fact that, in this case, we already had the information in hand to write down the proper integral for the length. In most cases this will not be the case and you will need to go back and do a shortened version of the analysis we did in the Parametric Equations and Curves section. We don't need all the information but to get the pertinent information we will need to go through most of the analysis.

If you need a refresher on how to do that analysis you should go back to that section and work through a few of the practice problems.

Step 4

Okay, let's get to work on evaluating the integral. At first glance this looks like a really unpleasant integral (and there's no ignoring the fact that it's not a super easy integral) however if we're careful it isn't as difficult as it might appear at first glance.

First, let's notice that we can do a little simplification as follows,

$$L = \int_{-2\pi}^{2\pi} \sqrt{\cos^2\left(\tfrac{1}{4}t\right)\left(1 + \sin^2\left(\tfrac{1}{4}t\right)\right)}\,dt = \int_{-2\pi}^{2\pi} \left|\cos\left(\tfrac{1}{4}t\right)\right| \sqrt{1 + \sin^2\left(\tfrac{1}{4}t\right)}\,dt$$

As always, be very careful with the absolute value bars! Depending on the range of *t*'s we used for the integral it might not be possible to just drop them. So, the next thing let's do is determine how to deal with them.

We know that, for this integral, we used the following range of *t*'s.

$$-2\pi \le t \le 2\pi$$

Now, notice that we don't just have a *t* in the argument for the trig functions in our integral. In fact what we have is $\tfrac{1}{4}t$. So, we can see that as *t* ranges over $-2\pi \le t \le 2\pi$ we will get the following range for $\tfrac{1}{4}t$ (just divide the above inequality by 4!),

$$-\tfrac{\pi}{2} \le \tfrac{1}{4}t \le \tfrac{\pi}{2}$$

We know from our trig knowledge that as long as $\tfrac{1}{4}t$ stays in this range, which it will for our integral, then $\cos\left(\tfrac{1}{4}t\right) \ge 0$ and so we can drop the absolute values from the cosine in the integral.

As a final word of warning here keep in mind that for other ranges of *t*'s we might have had negative cosine in the range of *t*'s and so we'd need to add in a minus sign to the integrand when we dropped the absolute value bars. Whether or not we need to do this will depend upon the range of *t*'s we chose to use for one trace and so this quick analysis will need to be done for these kinds of integrals.

Step 5
Okay, at this point the integral for the length of the curve is now,

$$L = \int_{-2\pi}^{2\pi} \cos\left(\tfrac{1}{4}t\right)\sqrt{1+\sin^2\left(\tfrac{1}{4}t\right)}\,dt$$

This still looks to be an unpleasant integral. However, in this case note that we can use the following simply substitution to convert it into a relatively easy integral.

$$u = \sin\left(\tfrac{1}{4}t\right) \quad \to \quad \sin^2\left(\tfrac{1}{4}t\right) = u^2 \qquad du = \tfrac{1}{4}\cos\left(\tfrac{1}{4}t\right)dt$$

$$t = -2\pi: \quad u = \sin\left(-\tfrac{1}{2}\pi\right) = -1 \qquad\qquad t = 2\pi: \quad u = \sin\left(\tfrac{1}{2}\pi\right) = 1$$

Under this substitution the integral of the length of the curve is then,

$$L = \int_{-1}^{1} 4\sqrt{1+u^2}\,du$$

Step 6
Now, at this point we can see that we have a fairly simple trig substitution we'll need to evaluate to find the length of the curve. The trig substitution we'll need for this integral is,

$$t = \tan\theta \qquad dt = \sec^2\theta\,d\theta \qquad\qquad \sqrt{1+u^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \left|\sec\theta\right|$$

To get rid of the absolute value on the secant will need to convert the limits into θ limits.

$$u = -1: \qquad -1 = \tan\theta \quad \to \quad \tan\theta = -1 \quad \to \quad \theta = -\tfrac{\pi}{4}$$
$$u = 1: \qquad 1 = \tan\theta \quad \to \quad \tan\theta = 1 \quad \to \quad \theta = \tfrac{\pi}{4}$$

Okay, the corresponding range of θ for this problem is $-\tfrac{\pi}{4} \le \theta \le \tfrac{\pi}{4}$ (first and fourth quadrant) and in this range we know that secant is positive. Therefore the root becomes,

$$\sqrt{1+u^2} = \sec\theta$$

The length of the curve is then,

$$L = \int_{-2\pi}^{2\pi} \sqrt{\cos^2\left(\tfrac{1}{4}t\right)\left(1+\sin^2\left(\tfrac{1}{4}t\right)\right)}\,dt = \int_{-1}^{1} 4\sqrt{1+u^2}\,du$$
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 4\sec^3\theta\,d\theta = 2\left[\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\right]\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \boxed{9.1824}$$

Step 7
Okay, that was quite a bit of work to get the length of the curve. We now need to recall that we were also asked to get the total distance traveled by the particle. This is actually quite easy to get now that we have the length of the curve.

To get the total distance traveled all we need to recall is that we noted in Step 3 above that we determined in problem #8 from the Parametric Equations and Curves section that the curve will trace out 21.5 times. Since we also know the length of a single trace of the curve we know that the total distance traveled by the particle must be,

$$\text{Total Distance Travled} = (9.1824)(21.5) = \boxed{197.4216}$$

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4. Set up, but do not evaluate, an integral that gives the length of the parametric curve given by the following set of parametric equations. You may assume that the curve traces out exactly once for the given range of *t*'s.

$$x = 2 + t^2 \qquad y = \mathbf{e}^t \sin(2t) \qquad 0 \le t \le 3$$

Step 1
The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = \mathbf{e}^t\sin(2t) + 2\mathbf{e}^t\cos(2t)$$

Step 2
We'll need the *ds* for this problem.

$$ds = \sqrt{[2t]^2 + \left[\mathbf{e}^t\sin(2t)+2\mathbf{e}^t\cos(2t)\right]^2}\,dt = \sqrt{4t^2 + \left[\mathbf{e}^t\sin(2t)+2\mathbf{e}^t\cos(2t)\right]^2}\,dt$$

Step 3
The integral for the arc length is then,

$$L = \int ds = \int_0^3 \sqrt{4t^2 + \left[\mathbf{e}^t\sin(2t)+2\mathbf{e}^t\cos(2t)\right]^2}\,dt$$

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5. Set up, but do not evaluate, an integral that gives the length of the parametric curve given by the following set of parametric equations. You may assume that the curve traces out exactly once for the given range of *t*'s.

$$x = \cos^3(2t) \qquad y = \sin\left(1-t^2\right) \qquad -\tfrac{3}{2} \le t \le 0$$

Step 1

The first thing we'll need here are the following two derivatives.

$$\frac{dx}{dt} = -6\cos^2(2t)\sin(2t) \qquad\qquad \frac{dy}{dt} = -2t\cos\left(1-t^2\right)$$

Step 2

We'll need the *ds* for this problem.

$$\begin{aligned} ds &= \sqrt{\left[-6\cos^2(2t)\sin(2t)\right]^2 + \left[-2t\cos\left(1-t^2\right)\right]^2}\,dt \\ &= \sqrt{36\cos^4(2t)\sin^2(2t) + 4t^2\cos^2\left(1-t^2\right)}\,dt \end{aligned}$$

Step 3

The integral for the arc length is then,

$$L = \int ds = \int_{-\frac{3}{2}}^{0} \sqrt{36\cos^4(2t)\sin^2(2t) + 4t^2\cos^2\left(1-t^2\right)}\,dt$$