Section 4-6 : Integral Test

1. Determine if the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$$

Solution

There really isn't all that much to this problem. We could use the Integral Test on this series or we could just use the *p*-series Test we discussed in the notes for this section.

We can clearly see that $p = \pi > 1$ and so by the *p*-series Test this series must **converge.**

---

2. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{2}{3+5n}$$

Step 1

Okay, prior to using the Integral Test on this series we first need to verify that we can in fact use the Integral Test!

Step 2

The series terms are,

$$a_n = \frac{2}{3+5n}$$

We can clearly see that for the range of *n* in the series the terms are positive and so that condition is met.

Step 3

In this case because there is only one *n* in the denominator and because all the terms in the denominator are positive it is (hopefully) clear that,

$$a_n = \frac{2}{3+5n} > \frac{2}{3+5(n+1)} = a_{n+1}$$

and so the series terms are decreasing.

Okay, we now know that both of the conditions required for us to use the Integral Test have been verified we can proceed with the Integral Test.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 4
Now, let's compute the integral for the test.

$$\int_0^\infty \frac{2}{3+5x}\,dx = \lim_{t\to\infty}\int_0^t \frac{2}{3+5x}\,dx = \lim_{t\to\infty}\left(\frac{2}{5}\ln\left|3+5x\right|\right)\Bigg|_0^t = \lim_{t\to\infty}\left(\frac{2}{5}\ln\left|3+5t\right| - \frac{2}{5}\ln\left|3\right|\right) = \infty$$

Step 5
Okay, the integral from the last step is a divergent integral and so by the Integral Test the series must also be a **divergent** series.

---

3. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty}\frac{1}{\left(2n+7\right)^3}$$

Step 1
Okay, prior to using the Integral Test on this series we first need to verify that we can in fact use the Integral Test!

Step 2
The series terms are,

$$a_n = \frac{1}{\left(2n+7\right)^3}$$

We can clearly see that for the range of *n* in the series the terms are positive and so that condition is met.

Step 3
In this case because there is only one *n* in the denominator and because all the terms in the denominator are positive it is (hopefully) clear that,

$$a_n = \frac{1}{\left(2n+7\right)^3} > \frac{1}{\left(2\left(n+1\right)+7\right)^3} = a_{n+1}$$

and so the series terms are decreasing.

Okay, we now know that both of the conditions required for us to use the Integral Test have been verified we can proceed with the Integral Test.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 4
Now, let's compute the integral for the test.

$$\int_2^\infty \frac{1}{(2x+7)^3}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{(2x+7)^3}\,dx = \lim_{t\to\infty}\left.\left(-\frac{1}{4}\frac{1}{(2x+7)^2}\right)\right|_2^t$$

$$= \lim_{t\to\infty}\left(-\frac{1}{4}\frac{1}{(2t+7)^2}+\frac{1}{4}\frac{1}{(11)^2}\right) = \frac{1}{484}$$

Step 5
Okay, the integral from the last step is a convergent integral and so by the Integral Test the series must also be a **convergent** series.

---

4. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty}\frac{n^2}{n^3+1}$$

Step 1
Okay, prior to using the Integral Test on this series we first need to verify that we can in fact use the Integral Test!

Step 2
The series terms are,

$$a_n = \frac{n^2}{n^3+1}$$

We can clearly see that for the range of *n* in the series the terms are positive and so that condition is met.

Step 3
In this case we need to be a little more careful with checking the decreasing condition. We can't just plug in $n + 1$ into the series term as we've done in the first couple of problems in this section. Doing that would suggest that both the numerator and denominator will increase and so it's not all that clear cut of a case that the terms will be decreasing.

Therefore, we'll need to do a quick Calculus I increasing/decreasing analysis. Here the function for the series terms and its derivative.

$$f(x) = \frac{x^2}{x^3+1} \qquad\qquad f'(x) = \frac{2x - x^4}{\left(x^3+1\right)^2} = \frac{x\left(2-x^3\right)}{\left(x^3+1\right)^2}$$

With a quick number line or sign chart we can see that the function will increase for $0 < x < \sqrt[3]{2} = 1.2599$ and will decrease for $\sqrt[3]{2} = 1.2599 < x < \infty$. Because the function and series terms are the same we know that the series terms will have the same increasing/decreasing behavior.

So, from this analysis we can see that the series terms are not always decreasing but will be decreasing for $n > \sqrt[3]{2}$ which is sufficient for us to use to say that this condition is also met.

Okay, we now know that both of the conditions required for us to use the Integral Test have been verified we can proceed with the Integral Test.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 4
Now, let's compute the integral for the test.

$$\int_0^\infty \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty}\int_0^t \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty}\left(\left.\frac{1}{3}\ln\left|x^3+1\right|\right)\right|_0^t = \lim_{t\to\infty}\left(\frac{1}{3}\ln\left|t^3+1\right| - \ln\left(1\right)\right) = \infty$$

Step 5
Okay, the integral from the last step is a divergent integral and so by the Integral Test the series must also be a **divergent** series.

---

5. Determine if the following series converges or diverges.

$$\sum_{n=3}^{\infty} \frac{3}{n^2 - 3n + 2}$$

Step 1
Okay, prior to using the Integral Test on this series we first need to verify that we can in fact use the Integral Test!

Step 2
The series terms are,

$$a_n = \frac{3}{n^2 - 3n + 2}$$

We can clearly see that for $n \ge 3$ (which matches our range of *n* for the series) we will have,

$$n^2 \ge 3n \qquad \Rightarrow \qquad n^2 - 3n \ge 0 \qquad \Rightarrow \qquad n^2 - 3n + 2 \ge n^2 - 3n \ge 0$$

Therefore, the series terms are positive and so that condition is met.

Note that on occasion we'll need to do more than just state that the series terms are positive by inspection and do a little work to show that the terms really are positive!

Step 3
In this case we need to be a little more careful with checking the decreasing condition. We can't just plug in *n* + 1 into the series term as we've done in the first couple of problems in this section.

Doing that the first term in the denominator would be getting larger which would suggest the series term is decreasing. However, because the second term in the denominator is subtracted off if we increase *n* that would suggest the denominator is getting larger and hence the series term is increasing.

Because we have these "competing" interests we'll need to do a quick Calculus I increasing/decreasing analysis. Here the function for the series terms and its derivative.

$$f(x) = \frac{3}{x^2 - 3x + 2} \qquad\qquad f'(x) = \frac{9 - 6x}{\left(x^2 - 3x + 2\right)^2}$$

With a quick number line or sign chart we can see that the function will increase for $x < \frac{3}{2}$ and will decrease for $x > \frac{3}{2}$. Because the function and series terms are the same we know that the series terms will have the same increasing/decreasing behavior.

So, from this analysis we can see that the series terms are always decreasing for the range *n* in our series and so this condition is also met.

Okay, we now know that both of the conditions required for us to use the Integral Test have been verified we can proceed with the Integral Test.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 4
Now, let's compute the integral for the test. The integral we'll need to compute is,

$$\int_3^\infty \frac{3}{x^2-3x+2}\,dx$$

This integral will however require us to do some quick partial fractions in order to do the evaluation. Here is that quick work.

$$\frac{3}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \qquad \to \qquad 3 = A(x-2) + B(x-1)$$

$$\begin{aligned} x = 1 \quad 3 &= -A \\ x = 2 \quad 3 &= B \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} A &= -3 \\ B &= 3 \end{aligned}$$

The integral is then,

$$\begin{aligned}
\int_3^\infty \frac{3}{x-2} - \frac{3}{x-1}\,dx &= \lim_{t\to\infty}\int_3^t \frac{3}{x-2} - \frac{3}{x-1}\,dx = \lim_{t\to\infty}\left(3\ln|x-2| - 3\ln|x-1|\right)\Big|_3^t \\
&= \lim_{t\to\infty}\left[3\ln|t-2| - 3\ln|t-1| - \left(3\ln|1| - 3\ln|2|\right)\right] \\
&= \lim_{t\to\infty}\left[3\ln\left|\frac{t-2}{t-1}\right| + 3\ln|2|\right] = 3\ln\left(\frac{1}{1}\right) + 3\ln(2) = 3\ln(2)
\end{aligned}$$

Be careful with the limit of the first two terms! To correctly compute the limit they need to be combined using logarithm properties as shown and we can then do a L'Hospital's Rule on the argument of the log to compute the limit.

Step 5
Okay, the integral from the last step is a convergent integral and so by the Integral Test the series must also be a **convergent** series.