Section 1-1 : Integration by Parts

1. Evaluate $\int 4x\cos(2-3x)\,dx$.

Hint : Remember that we want to pick *u* and *dv* so that upon computing *du* and *v* and plugging everything into the Integration by Parts formula the new integral is one that we can do.

Step 1
The first step here is to pick *u* and *dv*. We want to choose *u* and *dv* so that when we compute *du* and *v* and plugging everything into the Integration by Parts formula the new integral we get is one that we can do.

With that in mind it looks like the following choices for *u* and *dv* should work for us.

$$u = 4x \qquad\qquad dv = \cos(2-3x)\,dx$$

Step 2
Next, we need to compute *du* (by differentiating *u*) and *v* (by integrating *dv*).

$$\begin{aligned} u &= 4x & \rightarrow \quad & du = 4dx \\ dv &= \cos(2-3x)\,dx & \rightarrow \quad & v = -\tfrac{1}{3}\sin(2-3x) \end{aligned}$$

Step 3
Plugging *u, du, v* and *dv* into the Integration by Parts formula gives,

$$\begin{aligned} \int 4x\cos(2-3x)\,dx &= (4x)\left(-\tfrac{1}{3}\sin(2-3x)\right) - \int -\tfrac{4}{3}\sin(2-3x)\,dx \\ &= -\tfrac{4}{3}x\sin(2-3x) + \tfrac{4}{3}\int \sin(2-3x)\,dx \end{aligned}$$

Step 4
Okay, the new integral we get is easily doable and so all we need to do to finish this problem out is do the integral.

$$\int 4x\cos(2-3x)\,dx = \boxed{-\tfrac{4}{3}x\sin(2-3x) + \tfrac{4}{9}\cos(2-3x) + c}$$

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2. Evaluate $\int_6^0 (2+5x)\mathbf{e}^{\frac{1}{3}x}\,dx$.

Hint : Remember that we want to pick *u* and *dv* so that upon computing *du* and *v* and plugging everything into the Integration by Parts formula the new integral is one that we can do.

Also, don't forget that the limits on the integral won't have any effect on the choices of *u* and *dv*.

Step 1
The first step here is to pick *u* and *dv*. We want to choose *u* and *dv* so that when we compute *du* and *v* and plugging everything into the Integration by Parts formula the new integral we get is one that we can do.

With that in mind it looks like the following choices for *u* and *dv* should work for us.

$$u = 2 + 5x \qquad\qquad dv = \mathbf{e}^{\frac{1}{3}x}\,dx$$

Step 2
Next, we need to compute *du* (by differentiating *u*) and *v* (by integrating *dv*).

$$\begin{aligned} u &= 2 + 5x & &\rightarrow & du &= 5dx \\ dv &= \mathbf{e}^{\frac{1}{3}x}\,dx & &\rightarrow & v &= 3\mathbf{e}^{\frac{1}{3}x} \end{aligned}$$

Step 3
We can deal with the limits as we do the integral or we can just do the indefinite integral and then take care of the limits in the last step. We will be using the later way of dealing with the limits for this problem.

So, plugging *u, du, v* and *dv* into the Integration by Parts formula gives,

$$\int (2+5x)\mathbf{e}^{\frac{1}{3}x} = (2+5x)\left(3\mathbf{e}^{\frac{1}{3}x}\right) - \int 5\left(3\mathbf{e}^{\frac{1}{3}x}\right)dx = 3\mathbf{e}^{\frac{1}{3}x}(2+5x) - 15\int \mathbf{e}^{\frac{1}{3}x}\,dx$$

Step 4
Okay, the new integral we get is easily doable so let's evaluate it to get,

$$\int (2+5x)\mathbf{e}^{\frac{1}{3}x} = 3\mathbf{e}^{\frac{1}{3}x}(2+5x) - 45\mathbf{e}^{\frac{1}{3}x} + c = 15x\mathbf{e}^{\frac{1}{3}x} - 39\mathbf{e}^{\frac{1}{3}x} + c$$

Step 5
The final step is then to take care of the limits.

$$\int_6^0 (2+5x)\mathbf{e}^{\frac{1}{3}x}\,dx = \left(15x\mathbf{e}^{\frac{1}{3}x} - 39\mathbf{e}^{\frac{1}{3}x}\right)\Big|_6^0 = \boxed{-39 - 51\mathbf{e}^2 = -415.8419}$$

Do not get excited about the fact that the lower limit is larger than the upper limit. This can happen on occasion and in no way affects how the integral is evaluated.

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3. Evaluate $\int \left(3t+t^2\right)\sin\left(2t\right)dt$.

Hint : Remember that we want to pick *u* and *dv* so that upon computing *du* and *v* and plugging everything into the Integration by Parts formula the new integral is one that we can do (or at least will be easier to deal with).

Step 1
The first step here is to pick *u* and *dv*. We want to choose *u* and *dv* so that when we compute *du* and *v* and plugging everything into the Integration by Parts formula the new integral we get is one that we can do or will at least be an integral that will be easier to deal with.

With that in mind it looks like the following choices for *u* and *dv* should work for us.

$$u = 3t + t^2 \qquad\qquad dv = \sin\left(2t\right)dt$$

Step 2
Next, we need to compute *du* (by differentiating *u*) and *v* (by integrating *dv*).

$$\begin{aligned} u &= 3t + t^2 & &\rightarrow & du &= \left(3+2t\right)dt \\ dv &= \sin\left(2t\right)dt & &\rightarrow & v &= -\tfrac{1}{2}\cos\left(2t\right) \end{aligned}$$

Step 3
Plugging *u, du, v* and *dv* into the Integration by Parts formula gives,

$$\int \left(3t+t^2\right)\sin\left(2t\right)dt = -\tfrac{1}{2}\left(3t+t^2\right)\cos\left(2t\right) + \tfrac{1}{2}\int \left(3+2t\right)\cos\left(2t\right)dt$$

Step 4
Now, the new integral is still not one that we can do with only Calculus I techniques. However, it is one that we can do another integration by parts on and because the power on the *t*'s have gone down by one we are heading in the right direction.

So, here are the choices for *u* and *dv* for the new integral.

$$\begin{aligned} u &= 3 + 2t & &\rightarrow & du &= 2dt \\ dv &= \cos\left(2t\right)dt & &\rightarrow & v &= \tfrac{1}{2}\sin\left(2t\right) \end{aligned}$$

Step 5
Okay, all we need to do now is plug these new choices of *u* and *dv* into the new integral we got in Step 3 and finish the problem out.

$$\begin{aligned}\int\left(3t+t^2\right)\sin\left(2t\right)dt &= -\tfrac{1}{2}\left(3t+t^2\right)\cos\left(2t\right)+\tfrac{1}{2}\left[\tfrac{1}{2}\left(3+2t\right)\sin\left(2t\right)-\int\sin\left(2t\right)dt\right]\\ &= -\tfrac{1}{2}\left(3t+t^2\right)\cos\left(2t\right)+\tfrac{1}{2}\left[\tfrac{1}{2}\left(3+2t\right)\sin\left(2t\right)+\tfrac{1}{2}\cos\left(2t\right)\right]+c\\ &= \boxed{-\tfrac{1}{2}\left(3t+t^2\right)\cos\left(2t\right)+\tfrac{1}{4}\left(3+2t\right)\sin\left(2t\right)+\tfrac{1}{4}\cos\left(2t\right)+c}\end{aligned}$$

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4. Evaluate $\int 6\tan^{-1}\left(\frac{8}{w}\right)dw$.

Hint : Be careful with your choices of *u* and *dv* here. If you think about it there is really only one way that the choice can be made here and have the problem be workable.

Step 1
The first step here is to pick *u* and *dv*.

Note that if we choose the inverse tangent for *dv* the only way to get *v* is to integrate *dv* and so we would need to know the answer to get the answer and so that won't work for us. Therefore, the only real choice for the inverse tangent is to let it be *u*.

So, here are our choices for *u* and *dv*.

$$u = 6\tan^{-1}\left(\tfrac{8}{w}\right) \qquad\qquad dv = dw$$

Don't forget the *dw*! The differential *dw* still needs to be put into the *dv* even though there is nothing else left in the integral.

Step 2
Next, we need to compute *du* (by differentiating *u*) and *v* (by integrating *dv*).

$$\begin{aligned} u &= 6\tan^{-1}\left(\tfrac{8}{w}\right) &&\to && du = 6\frac{-\frac{8}{w^2}}{1+\left(\frac{8}{w}\right)^2}dw = 6\frac{-\frac{8}{w^2}}{1+\frac{64}{w^2}}dw\\ dv &= dw &&\to && v = w\end{aligned}$$

Step 3
In order to complete this problem we'll need to do some rewrite on *du* as follows,

$$du = \frac{-48}{w^2+64}dw$$

Plugging *u, du, v* and *dv* into the Integration by Parts formula gives,

$$\int 6\tan^{-1}\left(\tfrac{8}{w}\right)dw = 6w\tan^{-1}\left(\tfrac{8}{w}\right) + 48\int \frac{w}{w^2+64}\,dw$$

Step 4
Okay, the new integral we get is easily doable (with the substitution $u = 64 + w^2$) and so all we need to do to finish this problem out is do the integral.

$$\int 6\tan^{-1}\left(\tfrac{8}{w}\right)dw = \boxed{6w\tan^{-1}\left(\tfrac{8}{w}\right) + 24\ln\left|w^2+64\right| + c}$$

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5. Evaluate $\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz$.

Hint : This is one of the few integration by parts problems where either function can go on *u* and *dv*. Be careful however to not get locked into an endless cycle of integration by parts.

Step 1
The first step here is to pick *u* and *dv*.

In this case we can put the exponential in either the *u* or the *dv* and the cosine in the other. It is one of the few problems where the choice doesn't really matter.

For this problem well use the following choices for *u* and *dv*.

$$u = \cos\left(\tfrac{1}{4}z\right) \qquad\qquad dv = \mathbf{e}^{2z}\,dz$$

Step 2
Next, we need to compute *du* (by differentiating *u*) and *v* (by integrating *dv*).

$$\begin{aligned} u &= \cos\left(\tfrac{1}{4}z\right) & &\rightarrow & du &= -\tfrac{1}{4}\sin\left(\tfrac{1}{4}z\right)dz \\ dv &= \mathbf{e}^{2z}dz & &\rightarrow & v &= \tfrac{1}{2}\mathbf{e}^{2z} \end{aligned}$$

Step 3
Plugging *u, du, v* and *dv* into the Integration by Parts formula gives,

$$\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz = \tfrac{1}{2}\mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right) + \tfrac{1}{8}\int \mathbf{e}^{2z}\sin\left(\tfrac{1}{4}z\right)dz$$

Step 4
We'll now need to do integration by parts again and to do this we'll use the following choices.

$$u = \sin\left(\tfrac{1}{4}z\right) \qquad \to \qquad du = \tfrac{1}{4}\cos\left(\tfrac{1}{4}z\right)dz$$
$$dv = \mathbf{e}^{2z}dz \qquad \to \qquad v = \tfrac{1}{2}\mathbf{e}^{2z}$$

Step 5
Plugging these into the integral from Step 3 gives,

$$\begin{aligned}\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz &= \tfrac{1}{2}\mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right) + \tfrac{1}{8}\left[\tfrac{1}{2}\mathbf{e}^{2z}\sin\left(\tfrac{1}{4}z\right) - \tfrac{1}{8}\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz\right] \\ &= \tfrac{1}{2}\mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right) + \tfrac{1}{16}\mathbf{e}^{2z}\sin\left(\tfrac{1}{4}z\right) - \tfrac{1}{64}\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz\end{aligned}$$

Step 6
To finish this problem all we need to do is some basic algebraic manipulation to get the identical integrals on the same side of the equal sign.

$$\begin{aligned}\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz &= \tfrac{1}{2}\mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right) + \tfrac{1}{16}\mathbf{e}^{2z}\sin\left(\tfrac{1}{4}z\right) - \tfrac{1}{64}\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz \\ \int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz + \tfrac{1}{64}\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz &= \tfrac{1}{2}\mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right) + \tfrac{1}{16}\mathbf{e}^{2z}\sin\left(\tfrac{1}{4}z\right) \\ \tfrac{65}{64}\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz &= \tfrac{1}{2}\mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right) + \tfrac{1}{16}\mathbf{e}^{2z}\sin\left(\tfrac{1}{4}z\right)\end{aligned}$$

Finally, all we need to do is move the coefficient on the integral over to the right side.

$$\int \mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right)dz = \boxed{\tfrac{32}{65}\mathbf{e}^{2z}\cos\left(\tfrac{1}{4}z\right) + \tfrac{4}{65}\mathbf{e}^{2z}\sin\left(\tfrac{1}{4}z\right) + c}$$

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6. Evaluate $\int_0^{\pi} x^2\cos(4x)\,dx$.

Hint : Remember that we want to pick *u* and *dv* so that upon computing *du* and *v* and plugging everything into the Integration by Parts formula the new integral is one that we can do (or at least will be easier to deal with).

Also, don't forget that the limits on the integral won't have any effect on the choices of *u* and *dv*.

Step 1
The first step here is to pick *u* and *dv*. We want to choose *u* and *dv* so that when we compute *du* and *v* and plugging everything into the Integration by Parts formula the new integral we get is one that we can do or will at least be an integral that will be easier to deal with.

With that in mind it looks like the following choices for *u* and *dv* should work for us.

$$u = x^2 \qquad\qquad dv = \cos(4x)\,dx$$

Step 2
Next, we need to compute *du* (by differentiating *u*) and *v* (by integrating *dv*).

$$\begin{aligned} u &= x^2 && \rightarrow && du = 2x\,dx \\ dv &= \cos(4x)\,dx && \rightarrow && v = \tfrac{1}{4}\sin(4x) \end{aligned}$$

Step 3
We can deal with the limits as we do the integral or we can just do the indefinite integral and then take care of the limits in the last step. We will be using the later way of dealing with the limits for this problem.

So, plugging *u, du, v* and *dv* into the Integration by Parts formula gives,

$$\int x^2 \cos(4x)\,dx = \tfrac{1}{4}x^2\sin(4x) - \tfrac{1}{2}\int x\sin(4x)\,dx$$

Step 4
Now, the new integral is still not one that we can do with only Calculus I techniques. However, it is one that we can do another integration by parts on and because the power on the *x*'s have gone down by one we are heading in the right direction.

So, here are the choices for *u* and *dv* for the new integral.

$$\begin{aligned} u &= x && \rightarrow && du = dx \\ dv &= \sin(4x)\,dx && \rightarrow && v = -\tfrac{1}{4}\cos(4x) \end{aligned}$$

Step 5
Okay, all we need to do now is plug these new choices of *u* and *dv* into the new integral we got in Step 3 and evaluate the integral.

$$\begin{aligned} \int x^2 \cos(4x)\,dx &= \tfrac{1}{4}x^2\sin(4x) - \tfrac{1}{2}\left[-\tfrac{1}{4}x\cos(4x) + \tfrac{1}{4}\int\cos(4x)\,dx\right] \\ &= \tfrac{1}{4}x^2\sin(4x) - \tfrac{1}{2}\left[-\tfrac{1}{4}x\cos(4x) + \tfrac{1}{16}\sin(4x)\right] + c \\ &= \tfrac{1}{4}x^2\sin(4x) + \tfrac{1}{8}x\cos(4x) - \tfrac{1}{32}\sin(4x) + c \end{aligned}$$

Step 6
The final step is then to take care of the limits.

$$\int_0^{\pi} x^2\cos(4x)\,dx = \left(\tfrac{1}{4}x^2\sin(4x) + \tfrac{1}{8}x\cos(4x) - \tfrac{1}{32}\sin(4x)\right)\Bigg|_0^{\pi} = \boxed{\tfrac{1}{8}\pi}$$

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7. Evaluate $\int t^7 \sin\left(2t^4\right) dt$.

Hint : Be very careful with your choices of *u* and *dv* for this problem. It looks a lot like previous practice problems but it isn't!

Step 1

The first step here is to pick *u* and *dv* and, in this case, we'll need to be careful how we chose them.

If we follow the model of many of the examples/practice problems to this point it is tempting to let *u* be t^7 and to let *dv* be $\sin\left(2t^4\right)$.

However, this will lead to some real problems. To compute *v* we'd have to integrate the sine and because of the t^4 in the argument this is not possible. In order to integrate the sine we would have to have a t^3 in the integrand as well in order to a substitution as shown below,

$$\int t^3 \sin\left(2t^4\right) dt = \tfrac{1}{8}\int \sin\left(w\right) dw = -\tfrac{1}{8}\cos\left(2t^4\right) + c \qquad w = 2t^4$$

Now, this may seem like a problem, but in fact it's not a problem for this particular integral. Notice that we actually have 7 *t*'s in the integral and there is no reason that we can't split them up as follows,

$$\int t^7 \sin\left(2t^4\right) dt = \int t^4\, t^3 \sin\left(2t^4\right) dt$$

After doing this we can now choose *u* and *dv* as follows,

$$u = t^4 \qquad\qquad dv = t^3 \sin\left(2t^4\right) dt$$

Step 2

Next, we need to compute *du* (by differentiating *u*) and *v* (by integrating *dv*).

$$\begin{aligned} u &= t^4 & &\rightarrow & du &= 4t^3 dt \\ dv &= t^3 \sin\left(2t^4\right) dt & &\rightarrow & v &= -\tfrac{1}{8}\cos\left(2t^4\right) \end{aligned}$$

Step 3

Plugging *u, du, v* and *dv* into the Integration by Parts formula gives,

$$\int t^7 \sin\left(2t^4\right) dt = -\tfrac{1}{8} t^4 \cos\left(2t^4\right) + \tfrac{1}{2}\int t^3 \cos\left(2t^4\right) dt$$

Step 4

At this point, notice that the new integral just requires the same Calculus I substitution that we used to find *v*. So, all we need to do is evaluate the new integral and we'll be done.

$$\int t^7 \sin\left(2t^4\right) dt = \boxed{-\tfrac{1}{8} t^4 \cos\left(2t^4\right) + \tfrac{1}{16} \sin\left(2t^4\right) + c}$$

Do not get so locked into patterns for these problems that you end up turning the patterns into "rules" on how certain kinds of problems work. Most of the easily seen patterns are also easily broken (as this problem has shown).

Because we (as instructors) tend to work a lot of "easy" problems initially they also tend to conform to the patterns that can be easily seen. This tends to lead students to the idea that the patterns will always work and then when they run into one where the pattern doesn't work they get in trouble. So, be careful!

Note as well that we're not saying that patterns don't exist and that it isn't useful to recognize them. You just need to be careful and understand that there will, on occasion, be problems where it will look like a pattern you recognize, but in fact will not quite fit the pattern and another approach will be needed to work the problem.

Alternate Solution

Note that there is an alternate solution to this problem. We could use the substitution $w = 2t^4$ as the first step as follows.

$$w = 2t^4 \qquad \to \qquad dw = 8t^3 dt \qquad \& \qquad t^4 = \tfrac{1}{2} w$$

$$\int t^7 \sin\left(2t^4\right) dt = \int t^4\, t^3 \sin\left(2t^4\right) dt = \int \left(\tfrac{1}{2} w\right)\left(\tfrac{1}{8}\right) \sin\left(w\right) dw = \int \tfrac{1}{16} w \sin\left(w\right) dw$$

We won't avoid integration by parts as we can see here, but it is somewhat easier to see it this time. Here is the rest of the work for this problem.

$$u = \tfrac{1}{16} w \qquad \to \qquad du = \tfrac{1}{16} dw$$
$$dv = \sin\left(w\right) dw \qquad \to \qquad v = -\cos\left(w\right)$$

$$\int t^7 \sin\left(2t^4\right) dt = -\tfrac{1}{16} w \cos\left(w\right) + \tfrac{1}{16} \int \cos\left(w\right) dw = -\tfrac{1}{16} w \cos\left(w\right) + \tfrac{1}{16} \sin\left(w\right) + c$$

As the final step we just need to substitution back in for *w*.

$$\int t^7 \sin\left(2t^4\right) dt = -\tfrac{1}{8} t^4 \cos\left(2t^4\right) + \tfrac{1}{16} \sin\left(2t^4\right) + c$$

8. Evaluate $\int y^6 \cos(3y)\,dy$.

Hint : Doing this with "standard" integration by parts would take a fair amount of time so maybe this would be a good candidate for the "table" method of integration by parts.

Step 1

Okay, with this problem doing the "standard" method of integration by parts (*i.e.* picking *u* and *dv* and using the formula) would take quite a bit of time. So, this looks like a good problem to use the table that we saw in the notes to shorten the process up.

Here is the table for this problem.

$$\begin{array}{rr|c} y^6 & \cos(3y) & + \\ 6y^5 & \frac{1}{3}\sin(3y) & - \\ 30y^4 & -\frac{1}{9}\cos(3y) & + \\ 120y^3 & -\frac{1}{27}\sin(3y) & - \\ 360y^2 & \frac{1}{81}\cos(3y) & + \\ 720y & \frac{1}{243}\sin(3y) & - \\ 720 & -\frac{1}{729}\cos(3y) & + \\ 0 & -\frac{1}{2187}\sin(3y) & - \end{array}$$

Step 2

Here's the integral for this problem,

$$\begin{aligned}\int y^6 \cos(3y)\,dy &= \left(y^6\right)\left(\tfrac{1}{3}\sin(3y)\right) - \left(6y^5\right)\left(-\tfrac{1}{9}\cos(3y)\right) + \left(30y^4\right)\left(-\tfrac{1}{27}\sin(3y)\right) \\ &\qquad - \left(120y^3\right)\left(\tfrac{1}{81}\cos(3y)\right) + \left(360y^2\right)\left(\tfrac{1}{243}\sin(3y)\right) \\ &\qquad - (720y)\left(-\tfrac{1}{729}\cos(3y)\right) + (720)\left(-\tfrac{1}{2187}\sin(3y)\right) + c \\ &= \boxed{\begin{array}{c}\tfrac{1}{3}y^6\sin(3y) + \tfrac{2}{3}y^5\cos(3y) - \tfrac{10}{9}y^4\sin(3y) - \tfrac{40}{27}y^3\cos(3y) \\ + \tfrac{40}{27}y^2\sin(3y) + \tfrac{80}{81}y\cos(3y) - \tfrac{80}{243}\sin(3y) + c\end{array}}\end{aligned}$$

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9. Evaluate $\int \left(4x^3 - 9x^2 + 7x + 3\right)\mathbf{e}^{-x}\,dx$.

Hint : Doing this with "standard" integration by parts would take a fair amount of time so maybe this would be a good candidate for the "table" method of integration by parts.

Step 1

Okay, with this problem doing the "standard" method of integration by parts (*i.e.* picking u and dv and using the formula) would take quite a bit of time. So, this looks like a good problem to use the table that we saw in the notes to shorten the process up.

Here is the table for this problem.

$$\begin{array}{rr|c} 4x^3-9x^2+7x+3 & \mathbf{e}^{-x} & + \\ 12x^2-18x+7 & -\mathbf{e}^{-x} & - \\ 24x-18 & \mathbf{e}^{-x} & + \\ 24 & -\mathbf{e}^{-x} & - \\ 0 & \mathbf{e}^{-x} & + \end{array}$$

Step 2

Here's the integral for this problem,

$$\begin{aligned} \int\left(4x^3-9x^2+7x+3\right)\mathbf{e}^{-x}\,dx &= \left(4x^3-9x^2+7x+3\right)\left(-\mathbf{e}^{-x}\right)-\left(12x^2-18x+7\right)\left(\mathbf{e}^{-x}\right) \\ &\quad +\left(24x-18\right)\left(-\mathbf{e}^{-x}\right)-\left(24\right)\left(\mathbf{e}^{-x}\right)+c \\ &= -\mathbf{e}^{-x}\left(4x^3-9x^2+7x+3\right)-\mathbf{e}^{-x}\left(12x^2-18x+7\right) \\ &\quad -\mathbf{e}^{-x}\left(24x-18\right)-24\mathbf{e}^{-x}+c \\ &= \boxed{-\mathbf{e}^{-x}\left(4x^3+3x^2+13x+16\right)+c} \end{aligned}$$