Section 4-7 : Comparison Test/Limit Comparison Test

1. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+1\right)^2$$

Step 1

First, the series terms are,

$$a_n=\left(\frac{1}{n^2}+1\right)^2$$

and it should pretty obvious in this case that they are positive and so we know that we can use the Comparison Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2

For most of the Comparison Test problems we usually guess the convergence and proceed from there. However, in this case it is hopefully clear that for any *n*,

$$\left(\frac{1}{n^2}+1\right)^2>\left(1\right)^2=1$$

Now, let's take a look at the following series,

$$\sum_{n=1}^{\infty}1$$

Because $\lim\limits_{n\to\infty}1=1\neq0$ we can see from the Divergence Test that this series will be divergent.

So we've found a divergent series with terms that are smaller than the original series terms. Therefore, by the Comparison Test the series in the problem statement must also be **divergent**.

As a final note for this problem notice that we didn't actually need to do a Comparison Test to arrive at this answer. We could have just used the Divergence Test from the beginning since,

$$\lim_{n\to\infty}\left(\frac{1}{n^2}+1\right)^2 = 1 \neq 0$$

This is something that you should always keep in mind with series convergence problems. The Divergence Test is a quick test that can, on occasion, be used to quickly determine that a series diverges and hence avoid a lot of the hassles of some of the other tests.

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2. Determine if the following series converges or diverges.

$$\sum_{n=4}^{\infty}\frac{n^2}{n^3-3}$$

Step 1
First, the series terms are,

$$a_n = \frac{n^2}{n^3-3}$$

and it should pretty obvious that as long as $n > \sqrt{3}$ (which we'll always have for this series) that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The "-3" in the denominator won't really affect the size of the denominator for large enough *n* and so it seems like for large *n* that the term will probably behave like,

$$b_n = \frac{n^2}{n^3} = \frac{1}{n}$$

We also know that the series,

$$\sum_{n=4}^{\infty} \frac{1}{n}$$

will diverge because it is a harmonic series or by the *p*-series Test.

Therefore, it makes some sense that we can guess the series in the problem statement will probably diverge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series diverges we'll need to find a series with smaller terms that we know, or can prove, diverges.

Note as well that we'll also need to prove that the new series terms really are smaller than the terms from the series in the problem statement. We can't just "hope" that the will be smaller.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms smaller by either making the numerator smaller or the denominator larger.

In this case it should be pretty clear that,

$$n^3 > n^3 - 3$$

Therefore, we'll have the following relationship.

$$\frac{n^2}{n^3} < \frac{n^2}{n^3 - 3}$$

You do agree with this right? The numerator in each is the same while the denominator in the left term is larger than the denominator in the right term. Therefore, the rational expression on the left must be smaller than the rational expression on the right.

Step 4
Now, the series,

$$\sum_{n=4}^{\infty} \frac{n^2}{n^3} = \sum_{n=4}^{\infty} \frac{1}{n}$$

is a divergent series (as discussed above) and we've also shown that the series terms in this series are smaller than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **diverge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

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3. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{7}{n(n+1)}$$

Step 1
First, the series terms are,

$$a_n = \frac{7}{n(n+1)}$$

and it should pretty obvious that for the range of n we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The "+1" in the denominator won't really affect the size of the denominator for large enough n and so it seems like for large n that the term will probably behave like,

$$b_n = \frac{7}{n(n)} = \frac{7}{n^2}$$

We also know that the series,

$$\sum_{n=2}^{\infty} \frac{7}{n^2}$$

will converge by the *p*-series Test ($p = 2 > 1$).

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

In this case it should be pretty clear that,

$$n < n+1 \qquad \Rightarrow \qquad n(n) < n(n+1)$$

Therefore, we'll have the following relationship.

$$\frac{7}{n(n)} > \frac{7}{n(n+1)}$$

You do agree with this right? The numerator in each is the same while the denominator in the left term is smaller than the denominator in the right term. Therefore, the rational expression on the left must be larger than the rational expression on the right.

Step 4
Now, the series,

$$\sum_{n=2}^{\infty} \frac{7}{n(n)} = \sum_{n=2}^{\infty} \frac{7}{n^2}$$

is a convergent series (as discussed above) and we've also shown that the series terms in this series are larger than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **converge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

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4. Determine if the following series converges or diverges.

$$\sum_{n=7}^{\infty} \frac{4}{n^2 - 2n - 3}$$

Step 1
First, the series terms are,

$$a_n = \frac{4}{n^2 - 2n - 3}$$

You can verify that for $n \geq 7$ we have $n^2 > 2n + 3$ and so $n^2 - 2n - 3 = n^2 - (2n + 3) > 0$. Therefore, the series terms are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

For large enough *n* we know that the n^2 (a quadratic term) in the denominator will increase at a much faster rate than the $-2n - 3$ (a linear term) portion of the denominator. Therefore the n^2 portion of the denominator will, in all likelihood, define the behavior of the denominator and so the terms should behave like,

$$b_n = \frac{4}{n^2}$$

We also know that the series,

$$\sum_{n=4}^{\infty} \frac{4}{n^2}$$

will converge by the *p*-series Test ($p = 2 > 1$).

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

We now have a problem however. The obvious thing to try is to drop the last two terms on the denominator. Doing that however gives the following inequality,

$$n^2 > n^2 - 2n - 3$$

This in turn gives the following relationship.

$$\frac{4}{n^2} < \frac{4}{n^2 - 2n - 3}$$

The denominator on the left is larger and so the rational expression on the left must be smaller. This leads to the problem. While the series,

$$\sum_{n=4}^{\infty} \frac{4}{n^2}$$

will definitely converge (as discussed above) it's terms are smaller than the series terms in the problem statement. Just because a series with smaller terms converges does not, in any way, imply a series with larger terms will also converge!

There are other manipulations we might try but they are all liable to run into similar issues or end up with new terms that we wouldn't be able to quickly prove convergence on.

Hint : So, if the Comparison Test won't easily work what else is there to do?

Step 4

So, the Comparison Test won't easily work in this case. That pretty much leaves the Limit Comparison Test to try. This test only requires positive terms (which we have) and a second series that we're pretty sure behaves like the series we want to know the convergence for. Note as well that, for the Limit Comparison Test, we don't care if the terms for the second series are larger or smaller than problem statement series terms.

If you think about it we already have exactly what we need. In Step 2 we used a second series to guess at the convergence of the problem statement series. The terms in the new series are positive (which we need) and we're pretty sure it behaves in the same manner as the problem statement series.

So, let's compute the limit we need for the Limit Comparison Test.

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left[a_n \frac{1}{b_n} \right] = \lim_{n\to\infty} \left[\frac{4}{n^2 - 2n - 3} \frac{n^2}{4} \right] = \lim_{n\to\infty} \left[\frac{n^2}{n^2 - 2n - 3} \right] = 1$$

Step 5

Okay. We now have $0 < c = 1 < \infty$, *i.e.* *c* is not zero or infinity and so by the Limit Comparison Test the two series must have the same convergence. We determined in Step 2 that the second series converges and so the series given in the problem statement must also **converge**.

Be careful with the Comparison Test. Too often students just try to "force" larger or smaller by just hoping that the second series terms has the correct relationship (*i.e.* larger or smaller as needed) to the problem series terms. The problem is that this often leads to an incorrect answer. Be careful to always prove the larger/smaller nature of the series terms and if you can't get a series term of the correct larger/smaller nature then you may need to resort to the Limit Comparison Test.

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5. Determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{n-1}{\sqrt{n^6 + 1}}$$

Step 1

First, the series terms are,

$$a_n = \frac{n-1}{\sqrt{n^6 + 1}}$$

and it should pretty obvious that for the range of *n* we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The "-1" in the numerator and the "+1" in the denominator won't really affect the size of the numerator and denominator respectively for large enough *n* and so it seems like for large *n* that the term will probably behave like,

$$b_n = \frac{n}{\sqrt{n^6}} = \frac{n}{n^3} = \frac{1}{n^2}$$

We also know that the series,

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

will converge by the *p*-series Test ($p = 2 > 1$).

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

In this case we can work with both the numerator and the denominator. Let's start with the numerator. It should be pretty clear that,

$$n > n - 1$$

Using this we can make the numerator larger to get the following relationship,

$$\frac{n-1}{\sqrt{n^6+1}} < \frac{n}{\sqrt{n^6+1}}$$

Now, in the denominator it again is hopefully clear that,

$$n^6 < n^6 + 1$$

Using this we can make the denominator smaller (and hence make the rational expression larger) to get,

$$\frac{n-1}{\sqrt{n^6+1}} < \frac{n}{\sqrt{n^6+1}} < \frac{n}{\sqrt{n^6}} = \frac{1}{n^2}$$

Step 4
Now, the series,

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

is a convergent series (as discussed above) and we've also shown that the series terms in this series are larger than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **converge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

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6. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2n^3+7}{n^4 \sin^2(n)}$$

Step 1
First, the series terms are,

$$a_n = \frac{2n^3+7}{n^4 \sin^2(n)}$$

and it should pretty obvious that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The "+7" in the numerator and the "$\sin^2(n)$" in the denominator won't really affect the size of the numerator and denominator respectively for large enough *n* and so it seems like for large *n* that the term will probably behave like,

$$b_n = \frac{2n^3}{n^4} = \frac{2}{n}$$

We also know that the series,

$$\sum_{n=1}^{\infty} \frac{2}{n}$$

will diverge because it is a harmonic series or by the *p*-series Test.

Therefore, it makes some sense that we can guess the series in the problem statement will probably diverge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series diverges we'll need to find a series with smaller terms that we know, or can prove, diverges.

Note as well that we'll also need to prove that the new series terms really are smaller than the terms from the series in the problem statement. We can't just "hope" that the will be smaller.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms smaller by either making the numerator smaller or the denominator larger.

In this case we can work with both the numerator and the denominator. Let's start with the numerator. It should be pretty clear that,

$$2n^3 < 2n^3 + 7$$

Using this we can make the numerator smaller to get the following relationship,

$$\frac{2n^3 + 7}{n^4 \sin^2\left(n\right)} > \frac{2n^3}{n^4 \sin^2\left(n\right)}$$

Now, we know that $0 \le \sin^2\left(n\right) \le 1$ and so in the denominator we can see that if we replace the $\sin^2\left(n\right)$ with its largest possible value we have,

$$n^4 \sin^2\left(n\right) < n^4\left(1\right) = n^4$$

Using this we can make the denominator larger (and hence make the rational expression smaller) to get,

$$\frac{2n^3 + 7}{n^4 \sin^2\left(n\right)} > \frac{2n^3}{n^4 \sin^2\left(n\right)} > \frac{2n^3}{n^4} = \frac{2}{n}$$

Step 4
Now, the series,

$$\sum_{n=1}^{\infty} \frac{2}{n}$$

is a divergent series (as discussed above) and we've also shown that the series terms in this series are smaller than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **diverge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

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7. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{2^n \sin^2\left(5n\right)}{4^n + \cos^2\left(n\right)}$$

Step 1
First, the series terms are,

$$a_n = \frac{2^n \sin^2(5n)}{4^n + \cos^2(n)}$$

and it should pretty obvious that for the range of *n* we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

The trig functions in the numerator and in the denominator won't really affect the size of the numerator and denominator for large enough *n* and so it seems like for large *n* that the term will probably behave like,

$$b_n = \frac{2^n}{4^n} = \left(\frac{2}{4}\right)^n = \left(\frac{1}{2}\right)^n$$

We also know that the series,

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

will converge because it is a geometric series with $r = \frac{1}{2} < 1$.

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

In this case we can work with both the numerator and the denominator. Let's start with the numerator. We know that $0 \le \sin^2(5n) \le 1$ and so replacing the $\sin^2(5n)$ in the numerator with the largest possible value we get,

$$2^n \sin^2(5n) < 2^n(1) = 2^n$$

Using this we can make the numerator larger to get the following relationship,

$$\frac{2^n \sin^2(5n)}{4^n + \cos^2(n)} < \frac{2^n}{4^n + \cos^2(n)}$$

Now, in the denominator we know that $0 \le \cos^2(n) \le 1$ and so replacing the $\cos^2(n)$ with the smallest possible value we get,

$$4^n + \cos^2(n) > 4^n + 0 = 4^n$$

Using this we can make the denominator smaller (and hence make the rational expression larger) to get,

$$\frac{2^n \sin^2(5n)}{4^n + \cos^2(n)} < \frac{2^n}{4^n + \cos^2(n)} < \frac{2^n}{4^n} = \left(\frac{1}{2}\right)^n$$

Step 4
Now, the series,

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

is a convergent series (as discussed above) and we've also shown that the series terms in this series are larger than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **converge**.

Be careful with these kinds of problems. The series we used in Step 2 to make the guess ended up being the same series we used in the Comparison Test and this will often be the case but it will not always be that way. On occasion the two series will be different.

8. Determine if the following series converges or diverges.

$$\sum_{n=3}^{\infty} \frac{\mathbf{e}^{-n}}{n^2 + 2n}$$

Step 1
First, the series terms are,

$$a_n = \frac{\mathbf{e}^{-n}}{n^2 + 2n}$$

and it should pretty obvious that for the range of *n* we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
In this case let's first notice the exponential in the numerator will go to zero as *n* goes to infinity. Let's also notice that the denominator is just a polynomial. In cases like this the exponential is going to go to zero so fast that behavior of the denominator will not matter at all and in all probability the series in the problem statement will probably converge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

In this case we can work with both the numerator and the denominator. Let's start with the numerator. We can use some quick Calculus I to prove that $\mathbf{e}^{-n}$ is a decreasing function and so,

$$\mathbf{e}^{-n} < \mathbf{e}^{-3} < 1$$

Using this we can make the numerator larger to get the following relationship,

$$\frac{\mathbf{e}^{-n}}{n^2 + 2n} < \frac{1}{n^2 + 2n}$$

Now, in the denominator it should be fairly clear that,

$$n^2 + 2n > n^2$$

Using this we can make the denominator smaller (and hence make the rational expression larger) to get,

$$\frac{\mathbf{e}^{-n}}{n^2 + 2n} < \frac{1}{n^2 + 2n} < \frac{1}{n^2}$$

Step 4
Now, the series,

$$\sum_{n=3}^{\infty} \frac{1}{n^2}$$

is a convergent series (*p*-series Test with $p = 2 > 1$) and we've also shown that the series terms in this series are larger than the series terms from the series in the problem statement.

Therefore, by the Comparison Test the series given in the problem statement must also **converge**.

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9. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{4n^2 - n}{n^3 + 9}$$

Step 1
First, the series terms are,

$$a_n = \frac{4n^2 - n}{n^3 + 9}$$

You can verify that for $n \geq 1$ we have $4n^2 > n$ and so $4n^2 - n > 0$. Therefore, the series terms are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2
Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

For large enough *n* we know that the n^2 (a quadratic term) in the numerator will increase at a much faster rate than the $-n$ (a linear term) portion of the numerator. Therefore the n^2 portion of the numerator will, in all likelihood, define the behavior of the numerator. Likewise, the "+9" in the denominator will not affect the size of the denominator for large *n* and so the terms should behave like,

$$b_n = \frac{4n^2}{n^3} = \frac{4}{n}$$

We also know that the series,

$$\sum_{n=1}^{\infty} \frac{4}{n}$$

will diverge because it is a harmonic series or by the *p*-series Test.

Therefore, it makes some sense that we can guess the series in the problem statement will probably diverge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series diverge we'll need to find a series with smaller terms that we know, or can prove, diverge.

Note as well that we'll also need to prove that the new series terms really are smaller than the terms from the series in the problem statement. We can't just "hope" that the will be smaller.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms smaller by either making the numerator smaller or the denominator larger.

We now have a problem however. The obvious thing to try is to drop the last term in both the numerator and the denominator. Doing that however gives the following inequalities,

$$4n^2 - n < 4n^2 \qquad\qquad n^3 + 9 > n^3$$

Using these two in the series terms gives the following relationship,

$$\frac{4n^2 - n}{n^3 + 9} < \frac{4n^2}{n^3 + 9} < \frac{4n^2}{n^3} = \frac{4}{n}$$

Now the series,

$$\sum_{n=0}^{\infty} \frac{4}{n}$$

will definitely diverge (as discussed above) it's terms are larger than the series terms in the problem statement. Just because a series with larger terms diverges does not, in any way, imply a series with smaller terms will also diverge!

There are other manipulations we might try but they are all liable to run into similar issues or end up with new terms that we wouldn't be able to quickly prove convergence on.

Hint : So, if the Comparison Test won't easily work what else is there to do?

Step 4

So, the Comparison Test won't easily work in this case. That pretty much leaves the Limit Comparison Test to try. This test only requires positive terms (which we have) and a second series that we're pretty sure behaves like the series we want to know the convergence for. Note as well that, for the Limit Comparison Test, we don't care if the terms for the second series are larger or smaller than problem statement series terms.

If you think about it we already have exactly what we need. In Step 2 we used a second series to guess at the convergence of the problem statement series. The terms in the new series are positive (which we need) and we're pretty sure it behaves in the same manner as the problem statement series.

So, let's compute the limit we need for the Limit Comparison Test.

$$c = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left[a_n \frac{1}{b_n} \right] = \lim_{n \to \infty} \left[\frac{4n^2 - n}{n^3 + 9} \, \frac{n}{4} \right] = \lim_{n \to \infty} \left[\frac{4n^3 - n^2}{4n^3 + 36} \right] = 1$$

Step 5

Okay. We now have $0 < c = 1 < \infty$, *i.e.* *c* is not zero or infinity and so by the Limit Comparison Test the two series must have the same convergence. We determined in Step 2 that the second series diverges and so the series given in the problem statement must also **diverge**.

Be careful with the Comparison Test. Too often students just try to "force" larger or smaller by just hoping that the second series terms has the correct relationship (*i.e.* larger or smaller as needed) to the problem series terms. The problem is that this often leads to an incorrect answer. Be careful to always prove the larger/smaller nature of the series terms and if you can't get a series term of the correct larger/smaller nature then you may need to resort to the Limit Comparison Test.

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10. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\sqrt{2n^2 + 4n + 1}}{n^3 + 9}$$

Step 1

First, the series terms are,

$$a_n = \frac{\sqrt{2n^2 + 4n + 1}}{n^3 + 9}$$

and it should pretty obvious that for the range of *n* we have in this series that they are positive and so we know that we can attempt the Comparison Test for this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Hint : Can you make a guess as to whether or not the series should converge or diverge?

Step 2

Let's first see if we can make a reasonable guess as to whether this series converges or diverges.

For large enough *n* we know that the $2n^2$ (a quadratic term) in the numerator will increase at a much faster rate than the $4n+1$ (a linear term) portion of the numerator. Therefore the $2n^2$ portion of the numerator will, in all likelihood, define the behavior of the numerator. Likewise, the "+9" in the denominator will not affect the size of the denominator for large *n* and so the terms should behave like,

$$b_n = \frac{\sqrt{2n^2}}{n^3} = \frac{\sqrt{2}}{n^2}$$

We also know that the series,

$$\sum_{n=1}^{\infty} \frac{\sqrt{2}}{n^2}$$

will converge by the *p*-series Test ($p = 2 > 1$).

Therefore, it makes some sense that we can guess the series in the problem statement will probably converge as well.

Hint : Now that we have our guess, if we're going to use the Comparison Test, do we need to find a series with larger or a smaller terms that has the same convergence/divergence?

Step 3
So, because we're guessing that the series converge we'll need to find a series with larger terms that we know, or can prove, converge.

Note as well that we'll also need to prove that the new series terms really are larger than the terms from the series in the problem statement. We can't just "hope" that the will be larger.

In this case, because the terms in the problem statement series are a rational expression, we know that we can make the series terms larger by either making the numerator larger or the denominator smaller.

We now have a problem however. The obvious thing to try is to drop the last two terms in the numerator and the last term in the denominator. Doing that however gives the following inequalities,

$$2n^2 < 2n^2 + 4n + 1 \qquad\qquad n^3 + 9 > n^3$$

This leads to a real problem! If we use the inequality for the numerator we're going to get a smaller rational expression and if we use the inequality for the denominator we're going to get a larger rational expression. Because these two can't both be used at the same time it will make it fairly difficult to use the Comparison Test since neither one individually give a series we can quickly deal with.

Hint : So, if the Comparison Test won't easily work what else is there to do?

Step 4
So, the Comparison Test won't easily work in this case. That pretty much leaves the Limit Comparison Test to try. This test only requires positive terms (which we have) and a second series that we're pretty sure behaves like the series we want to know the convergence for. Note as well that, for the Limit Comparison Test, we don't care if the terms for the second series are larger or smaller than problem statement series terms.

If you think about it we already have exactly what we need. In Step 2 we used a second series to guess at the convergence of the problem statement series. The terms in the new series are positive (which we need) and we're pretty sure it behaves in the same manner as the problem statement series.

So, let's compute the limit we need for the Limit Comparison Test.

$$c = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left[a_n \frac{1}{b_n} \right] = \lim_{n \to \infty} \left[\frac{\sqrt{2n^2 + 4n + 1}}{n^3 + 9} \frac{n^2}{\sqrt{2}} \right]$$

$$= \lim_{n \to \infty} \left[\frac{n^2 \sqrt{n^2 \left(2 + \frac{4}{n} + \frac{1}{n^2}\right)}}{\sqrt{2}\, n^3 \left(1 + \frac{9}{n^3}\right)} \right] = \lim_{n \to \infty} \left[\frac{n^2 (n) \sqrt{2 + \frac{4}{n} + \frac{1}{n^2}}}{\sqrt{2}\, n^3 \left(1 + \frac{9}{n^3}\right)} \right] = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

Step 5

Okay. We now have $0 < c = 1 < \infty$, *i.e.* *c* is not zero or infinity and so by the Limit Comparison Test the two series must have the same convergence. We determined in Step 2 that the second series converges and so the series given in the problem statement must also **converge**.

Be careful with the Comparison Test. Too often students just try to "force" larger or smaller by just hoping that the second series terms has the correct relationship (*i.e.* larger or smaller as needed) to the problem series terms. The problem is that this often leads to an incorrect answer. Be careful to always prove the larger/smaller nature of the series terms and if you can't get a series term of the correct larger/smaller nature then you may need to resort to the Limit Comparison Test.