Section 4-8 : Alternating Series Test

1. Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{7+2n}$$

Step 1
First, this is (hopefully) clearly an alternating series with,

$$b_n = \frac{1}{7+2n}$$

and it should pretty obvious the b_n are positive and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2
Let's first take a look at the limit,

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{7+2n} = 0$$

So, the limit is zero and so the first condition is met.

Step 3
Now let's take care of the decreasing check. In this case it should be pretty clear that,

$$\frac{1}{7+2n} > \frac{1}{7+2(n+1)}$$

since increasing *n* will only increase the denominator and hence force the rational expression to be smaller.

Therefore the b_n form a decreasing sequence.

Step 4
So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

2. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+3}}{n^3+4n+1}$$

Step 1
First, this is (hopefully) clearly an alternating series with,

$$b_n = \frac{1}{n^3+4n+1}$$

and it should pretty obvious the b_n are positive and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2
Let's first take a look at the limit,

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n^3+4n+1} = 0$$

So, the limit is zero and so the first condition is met.

Step 3
Now let's take care of the decreasing check. In this case it should be pretty clear that,

$$\frac{1}{n^3+4n+1} > \frac{1}{(n+1)^3+4(n+1)+1}$$

since increasing *n* will only increase the denominator and hence force the rational expression to be smaller.

Therefore the b_n form a decreasing sequence.

Step 4
So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

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3. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{1}{(-1)^n (2^n + 3^n)}$$

Step 1

Do not get excited about the $(-1)^n$ is in the denominator! This is still an alternating series! All the $(-1)^n$ does is change the sign regardless of whether or not it is in the numerator.

Also note that we could just as easily rewrite the terms as,

$$\frac{1}{(-1)^n (2^n + 3^n)} = \frac{(-1)^n}{(-1)^n} \frac{1}{(-1)^n (2^n + 3^n)} = \frac{(-1)^n}{(-1)^{2n} (2^n + 3^n)} = \frac{(-1)^n}{(2^n + 3^n)}$$

Note that $(-1)^{2n} = 1$ because the exponent is always even!

So, we now know that this is an alternating series with,

$$b_n = \frac{1}{2^n + 3^n}$$

and it should pretty obvious the b_n are positive and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2

Let's first take a look at the limit,

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{2^n + 3^n} = 0$$

So, the limit is zero and so the first condition is met.

Step 3

Now let's take care of the decreasing check. In this case it should be pretty clear that,

$$\frac{1}{2^n + 3^n} > \frac{1}{2^{n+1} + 3^{n+1}}$$

since increasing *n* will only increase the denominator and hence force the rational expression to be smaller.

Therefore the b_n form a decreasing sequence.

Step 4
So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

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4. Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+6} n}{n^2+9}$$

Step 1
First, this is (hopefully) clearly an alternating series with,

$$b_n = \frac{n}{n^2+9}$$

and it should pretty obvious the b_n are positive and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2
Let's first take a look at the limit,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+9} = 0$$

So, the limit is zero and so the first condition is met.

Step 3
Now let's take care of the decreasing check. In this case increasing *n* will increase both the numerator and denominator and so we can't just say that clearly the terms are decreasing as we did in the first few problems.

We will have no choice but to do a little Calculus I work for this problem. Here is the function and derivative for that work.

$$f(x) = \frac{x}{x^2+9} \qquad f'(x) = \frac{9-x^2}{\left(x^2+9\right)^2}$$

It should be pretty clear that the function will be increasing in $0 \le x < 3$ and decreasing in $x > 3$ (the range of *x* that corresponds to our range of *n*).

So, the b_n do not actually form a decreasing sequence but they are decreasing for $n > 3$ and so we can say that they are eventually decreasing and as discussed in the notes that will be sufficient for us.

Step 4
So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.

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5. Determine if the following series converges or diverges.

$$\sum_{n=4}^{\infty} \frac{(-1)^{n+2}(1-n)}{3n-n^2}$$

Step 1
First, this is (hopefully) clearly an alternating series with,

$$b_n = \frac{1-n}{3n-n^2}$$

and b_n are positive for $n \ge 4$ and so we know that we can use the Alternating Series Test on this series.

It is very important to always check the conditions for a particular series test prior to actually using the test. One of the biggest mistakes that many students make with the series test is using a test on a series that don't meet the conditions for the test and getting the wrong answer because of that!

Step 2
Let's first take a look at the limit,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1-n}{3n-n^2} = 0$$

So, the limit is zero and so the first condition is met.

Step 3

Now let's take care of the decreasing check. In this case increasing *n* will increase both the numerator and denominator and so we can't just say that clearly the terms are decreasing as we did in the first few problems.

We will have no choice but to do a little Calculus I work for this problem. Here is the function and derivative for that work.

$$f\left(x\right)=\frac{1-x}{3x-x^2} \qquad\qquad f'\left(x\right)=\frac{-x^2+2x-3}{\left(3x-x^2\right)^2}$$

The numerator of the derivative is never zero for any real number (we'll leave that to you to verify) and since it is clearly negative at $x=0$ we know that the function will always be decreasing for $x\geq 4$.

Therefore the b_n form a decreasing sequence.

Step 4

So, both of the conditions in the Alternating Series Test are met and so the series is **convergent**.