Section 4-9 : Absolute Convergence

1. Determine if the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{\left(-1\right)^{n+1}}{n^3+1}$$

Step 1

Okay, let's first see if the series converges or diverges if we put absolute value on the series terms.

$$\sum_{n=2}^{\infty} \left| \frac{\left(-1\right)^{n+1}}{n^3+1} \right| = \sum_{n=2}^{\infty} \frac{1}{n^3+1}$$

Now, notice that,

$$\frac{1}{n^3+1} < \frac{1}{n^3}$$

and we know by the *p*-series test that

$$\sum_{n=2}^{\infty} \frac{1}{n^3}$$

converges.

Therefore, by the Comparison Test we know that the series from the problem statement,

$$\sum_{n=2}^{\infty} \frac{1}{n^3+1}$$

will also converge.

Step 2

So, because the series with the absolute value converges we know that the series in the problem statement is **absolutely convergent**.

---

2. Determine if the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\left(-1\right)^{n-3}}{\sqrt{n}}$$

Step 1

Okay, let's first see if the series converges or diverges if we put absolute value on the series terms.

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-3}}{\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}$$

Now, by the by the *p*-series test we can see that this series will diverge.

Step 2
So, at this point we know that the series in the problem statement is not absolutely convergent so all we need to do is check to see if it's conditionally convergent or divergent. To do this all we need to do is check the convergence of the series in the problem statement.

The series in the problem statement is an alternating series with,

$$b_n = \frac{1}{\sqrt{n}}$$

Clearly the b_n are positive so we can use the Alternating Series Test on this series. It is hopefully clear that the b_n are a decreasing sequence and $\lim_{n\to\infty} b_n = 0$.

Therefore, by the Alternating Series Test the series from the problem statement is convergent.

Step 3
So, because the series with the absolute value diverges and the series from the problem statement converges we know that the series in the problem statement is **conditionally convergent**.

---

3. Determine if the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=3}^{\infty}\frac{(-1)^{n+1}(n+1)}{n^3+1}$$

Step 1
Okay, let's first see if the series converges or diverges if we put absolute value on the series terms.

$$\sum_{n=3}^{\infty}\left|\frac{(-1)^{n+1}(n+1)}{n^3+1}\right| = \sum_{n=3}^{\infty}\frac{n+1}{n^3+1}$$

We know by the *p*-series test that the following series converges.

$$\sum_{n=3}^{\infty} \frac{1}{n^2}$$

If we now compute the following limit,

$$c = \lim_{n \to \infty} \left[\frac{n+1}{n^3+1} \frac{n^2}{1} \right] = \lim_{n \to \infty} \left[\frac{n^3+n^2}{n^3+1} \right] = 1$$

we know by the Limit Comparison Test that the two series in this problem have the same convergence because *c* is neither zero or infinity and because $\sum_{n=3}^{\infty} \frac{1}{n^2}$ converges we know that the series from the problem statement must also converge.

Step 2
So, because the series with the absolute value converges we know that the series in the problem statement is **absolutely convergent**.
