Preface

Here are the solutions to the practice problems for my Calculus I notes. Some solutions will have more or less detail than other solutions. The level of detail in each solution will depend up on several issues. If the section is a review section, this mostly applies to problems in the first chapter, there will probably not be as much detail to the solutions given that the problems really should be review. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you've reached the level of working the harder problems then you will probably already understand the basics fairly well and won't need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven't been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

Substitution Rule for Definite Integrals

1. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_0^1 3\left(4x + x^4\right)\left(10x^2 + x^5 - 2\right)^6 dx$$

Step 1
The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

The substitution for this problem is,

$$u = 10x^2 + x^5 - 2$$

Step 2
Here is the actual substitution work for this problem.

$$du = \left(20x + 5x^4\right)dx = 5\left(4x + x^4\right)dx \qquad \rightarrow \qquad \left(4x + x^4\right)dx = \tfrac{1}{5}du$$

$$x = 0 : u = -2 \qquad\qquad x = 1 : u = 9$$

As we did in the notes for this section we are also going to convert the limits to u's to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_0^1 3\left(4x + x^4\right)\left(10x^2 + x^5 - 2\right)^6 dx = \tfrac{3}{5}\int_{-2}^9 u^6\, du$$

Step 3
The integral is then,

$$\int_0^1 3\left(4x + x^4\right)\left(10x^2 + x^5 - 2\right)^6 dx = \tfrac{3}{35}u^7\Big|_{-2}^9 = \tfrac{3}{35}\left(4,782,969 - (-128)\right) = \boxed{\tfrac{14,349,291}{35}}$$

Do not get excited about "messy" or "large" answers. They will happen on occasion so don't worry about them when the happen.

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2. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_0^{\frac{\pi}{4}} \frac{8\cos(2t)}{\sqrt{9 - 5\sin(2t)}}\, dt$$

Step 1
The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

The substitution for this problem is,
$$u = 9 - 5\sin(2t)$$

Step 2
Here is the actual substitution work for this problem.

$$du = -10\cos(2t)\,dt \qquad \rightarrow \qquad \cos(2t)\,dt = -\tfrac{1}{10}\,du$$
$$t = 0 : u = 9 \qquad\qquad t = \tfrac{\pi}{4} : u = 4$$

As we did in the notes for this section we are also going to convert the limits to u's to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_0^{\frac{\pi}{4}} \frac{8\cos(2t)}{\sqrt{9-5\sin(2t)}}\,dt = -\tfrac{8}{10}\int_9^4 u^{-\frac{1}{2}}\,du$$

Step 3
The integral is then,

$$\int_0^{\frac{\pi}{4}} \frac{8\cos(2t)}{\sqrt{9-5\sin(2t)}}\,dt = -\tfrac{8}{5}u^{\frac{1}{2}}\Big|_9^4 = -\tfrac{16}{5} - \left(-\tfrac{24}{5}\right) = \boxed{\tfrac{8}{5}}$$

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3. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_\pi^0 \sin(z)\cos^3(z)\,dz$$

Step 1
The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

The substitution for this problem is,
$$u = \cos(z)$$

Step 2
Here is the actual substitution work for this problem.

$$du = -\sin(z)\,dz \qquad \rightarrow \qquad \sin(z)\,dz = -du$$
$$z = \pi : u = -1 \qquad\qquad z = 0 : u = 1$$

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As we did in the notes for this section we are also going to convert the limits to u's to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_{\pi}^{0} \sin(z)\cos^3(z)\,dz = -\int_{-1}^{1} u^3\,du$$

Step 3
The integral is then,

$$\int_{\pi}^{0} \sin(z)\cos^3(z)\,dz = -\tfrac{1}{4}u^4\Big|_{-1}^{1} = -\tfrac{1}{4} - \left(-\tfrac{1}{4}\right) = \boxed{0}$$

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4. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{1}^{4} \sqrt{w}\,\mathbf{e}^{1-\sqrt{w^3}}\,dw$$

Step 1
The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

The substitution for this problem is,
$$u = 1 - w^{\frac{3}{2}}$$

Step 2
Here is the actual substitution work for this problem.

$$du = -\tfrac{3}{2}w^{\frac{1}{2}}dw \qquad \rightarrow \qquad \sqrt{w}\,dw = -\tfrac{2}{3}du$$
$$w = 1 : u = 0 \qquad\qquad\qquad w = 4 : u = -7$$

As we did in the notes for this section we are also going to convert the limits to u's to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_1^4 \sqrt{w}\, e^{1-\sqrt{w^3}}\, dw = -\tfrac{2}{3} \int_0^{-7} e^u\, du$$

Step 3
The integral is then,

$$\int_1^4 \sqrt{w}\, e^{1-\sqrt{w^3}}\, dw = -\tfrac{2}{3} e^u \Big|_0^{-7} = -\tfrac{2}{3} e^{-7} - \left(-\tfrac{2}{3} e^0\right) = \boxed{\tfrac{2}{3}\left(1 - e^{-7}\right)}$$

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5. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-4}^{-1} \sqrt[3]{5 - 2y} + \frac{7}{5 - 2y}\, dy$$

Step 1
The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

The substitution for this problem is,
$$u = 5 - 2y$$

Step 2
Here is the actual substitution work for this problem.

$$du = -2\, dy \qquad \rightarrow \qquad dy = -\tfrac{1}{2}\, du$$
$$y = -4 : u = 13 \qquad\qquad y = -1 : u = 7$$

As we did in the notes for this section we are also going to convert the limits to u's to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_{-4}^{-1} \sqrt[3]{5 - 2y} + \frac{7}{5 - 2y}\, dy = -\tfrac{1}{2} \int_{13}^{7} u^{\frac{1}{3}} + \frac{7}{u}\, du$$

Step 3
The integral is then,

$$\int_{-4}^{-1} \sqrt[3]{5-2y} + \frac{7}{5-2y} \, dy = \left(-\frac{1}{2}\left[\frac{3}{4}u^{\frac{4}{3}} + 7\ln|u|\right]\right)\Big|_{13}^{7}$$

$$= -\frac{3}{8}7^{\frac{4}{3}} - \frac{7}{2}\ln|7| - \left(-\frac{3}{8}13^{\frac{4}{3}} - \frac{7}{2}\ln|13|\right)$$

$$= \boxed{\frac{3}{8}\left(13^{\frac{4}{3}} - 7^{\frac{4}{3}}\right) + \frac{7}{2}\left(\ln(13) - \ln(7)\right)}$$

6. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-1}^{2} x^3 + e^{\frac{1}{4}x} \, dx$$

Step 1
The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

Before setting up the substitution we'll need to break up the integral because the first term doesn't need a substitution. Doing this gives,

$$\int_{-1}^{2} x^3 + e^{\frac{1}{4}x} \, dx = \int_{-1}^{2} x^3 \, dx + \int_{-1}^{2} e^{\frac{1}{4}x} \, dx$$

The substitution for the second integral is then,

$$u = \frac{1}{4}x$$

Step 2
Here is the actual substitution work for this second integral.

$$du = \frac{1}{4}dx \qquad \rightarrow \qquad dx = 4du$$
$$x = -1 : u = -\frac{1}{4} \qquad\qquad x = 2 : u = \frac{1}{2}$$

As we did in the notes for this section we are also going to convert the limits to u's to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_{-1}^{2} x^3 + e^{\frac{1}{4}x}\, dx = \int_{-1}^{2} x^3\, dx + 4\int_{-\frac{1}{4}}^{\frac{1}{2}} e^{u}\, du$$

Step 3
The integral is then,

$$\int_{-1}^{2} x^3 + e^{\frac{1}{4}x}\, dx = \tfrac{1}{4}x^4 \Big|_{-1}^{2} + 4e^{u}\Big|_{-\frac{1}{4}}^{\frac{1}{2}} = \left(4 - \tfrac{1}{4}\right) + \left(4e^{\frac{1}{2}} - 4e^{-\frac{1}{4}}\right) = \boxed{\tfrac{15}{4} + 4e^{\frac{1}{2}} - 4e^{-\frac{1}{4}}}$$

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7. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{\pi}^{\frac{3\pi}{2}} 6\sin(2w) - 7\cos(w)\, dw$$

Step 1
The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

Before setting up the substitution we'll need to break up the integral because the second term doesn't need a substitution. Doing this gives,

$$\int_{\pi}^{\frac{3\pi}{2}} 6\sin(2w) - 7\cos(w)\, dw = \int_{\pi}^{\frac{3\pi}{2}} 6\sin(2w)\, dw - \int_{\pi}^{\frac{3\pi}{2}} 7\cos(w)\, dw$$

The substitution for the first integral is then,

$$u = 2w$$

Step 2
Here is the actual substitution work for this first integral.

$$du = 2dw \qquad \rightarrow \qquad dw = \tfrac{1}{2}du$$
$$w = \pi : u = 2\pi \qquad\qquad w = \tfrac{3\pi}{2} : u = 3\pi$$

As we did in the notes for this section we are also going to convert the limits to u's to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_{\pi}^{\frac{3\pi}{2}} 6\sin(2w) - 7\cos(w)\, dw = 3\int_{2\pi}^{3\pi} \sin(u)\, du - \int_{\pi}^{\frac{3\pi}{2}} 7\cos(w)\, dw$$

Step 3
The integral is then,

$$\int_{\pi}^{\frac{3\pi}{2}} 6\sin(2w) - 7\cos(w)\, dw = -3\cos(u)\Big|_{2\pi}^{3\pi} - 7\sin(w)\Big|_{\pi}^{\frac{3\pi}{2}} = \left(3 - (-3)\right) + \left(7 - 0\right) = \boxed{13}$$

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8. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{1}^{5} \frac{2x^3 + x}{x^4 + x^2 + 1} - \frac{x}{x^2 - 4}\, dx$$

Solution

Be very careful with this problem. Recall that we can only do definite integrals if the integrand (*i.e.* the function we are integrating) is continuous on the interval over which we are integrating.

In this case the second term has division by zero at $x = 2$ and so is not continuous on $[1,5]$ and therefore this integral can't be done.

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9. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-2}^{0} t\sqrt{3 + t^2} + \frac{3}{(6t - 1)^2}\, dt$$

Step 1
The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

Before setting up the substitution we'll need to break up the integral because each term requires a different substitution. Doing this gives,

$$\int_{-2}^{0} t\sqrt{3+t^2} + \frac{3}{(6t-1)^2}\, dt = \int_{-2}^{0} t\sqrt{3+t^2}\, dt + \int_{-2}^{0} \frac{3}{(6t-1)^2}\, dt$$

The substitution for the each integral is then,

$$u = 3 + t^2 \qquad\qquad\qquad v = 6t - 1$$

Step 2
Here is the actual substitution work for this first integral.

$$du = 2t\, dt \qquad \rightarrow \qquad t\, dt = \tfrac{1}{2}\, du$$
$$t = -2 : u = 7 \qquad\qquad t = 0 : u = 3$$

Here is the actual substitution work for the second integral.

$$du = 6\, dt \qquad \rightarrow \qquad dt = \tfrac{1}{6}\, du$$
$$t = -2 : u = -13 \qquad\qquad t = 0 : u = -1$$

As we did in the notes for this section we are also going to convert the limits to u's to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_{-2}^{0} t\sqrt{3+t^2} + \frac{3}{(6t-1)^2}\, dt = \tfrac{1}{2}\int_{7}^{3} u^{\frac{1}{2}}\, du + \tfrac{3}{6}\int_{-13}^{-1} v^{-2}\, dv$$

Step 3
The integral is then,

$$\int_{-2}^{0} t\sqrt{3+t^2} + \frac{3}{(6t-1)^2}\, dt = \tfrac{1}{3}u^{\frac{3}{2}}\Big|_{7}^{3} - \tfrac{1}{2}v^{-1}\Big|_{-13}^{-1} = \tfrac{1}{3}\left(3^{\frac{3}{2}} - 7^{\frac{3}{2}}\right) - \tfrac{1}{2}\left(-1 - \left(-\tfrac{1}{13}\right)\right) = \boxed{\tfrac{1}{3}\left(3^{\frac{3}{2}} - 7^{\frac{3}{2}}\right) + \tfrac{6}{13}}$$

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10. Evaluate the following integral, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

$$\int_{-2}^{1} (2-z)^3 + \sin(\pi z)\big[3 + 2\cos(\pi z)\big]^3\, dz$$

Step 1
The first step that we need to do is do the substitution.

At this point you should be fairly comfortable with substitutions. If you are not comfortable with substitutions you should go back to the substitution sections and work some problems there.

Before setting up the substitution we'll need to break up the integral because each term requires a different substitution. Doing this gives,

$$\int_{-2}^{1}(2-z)^3 + \sin(\pi z)\left[3 + 2\cos(\pi z)\right]^3 dz = \int_{-2}^{1}(2-z)^3 dz + \int_{-2}^{1}\sin(\pi z)\left[3 + 2\cos(\pi z)\right]^3 dz$$

The substitution for the each integral is then,

$$u = 2 - z \qquad\qquad v = 3 + 2\cos(\pi z)$$

Step 2
Here is the actual substitution work for this first integral.

$$du = -dz \qquad \rightarrow \qquad dz = -du$$
$$z = -2 : u = 4 \qquad\qquad z = 1 : u = 1$$

Here is the actual substitution work for the second integral.

$$du = -2\pi \sin(\pi z)\,dz \qquad \rightarrow \qquad \sin(\pi z)\,dz = -\tfrac{1}{2\pi}\,du$$
$$z = -2 : u = 5 \qquad\qquad z = 1 : u = 1$$

As we did in the notes for this section we are also going to convert the limits to u's to avoid having to deal with the back substitution after doing the integral.

Here is the integral after the substitution.

$$\int_{-2}^{1}(2-z)^3 + \sin(\pi z)\left[3 + 2\cos(\pi z)\right]^3 dz = -\int_{4}^{1}u^3\,du - \tfrac{1}{2\pi}\int_{5}^{1}v^3\,dv$$

Step 3
The integral is then,

$$\int_{-2}^{1}(2-z)^3 + \sin(\pi z)\left[3 + 2\cos(\pi z)\right]^3 dz = -\tfrac{1}{4}u^4\Big|_{4}^{1} - \tfrac{1}{8\pi}v^4\Big|_{5}^{1}$$
$$= -\tfrac{1}{4}(1 - 256) - \tfrac{1}{8\pi}(1 - 625) = \boxed{\tfrac{255}{4} + \tfrac{78}{\pi}}$$