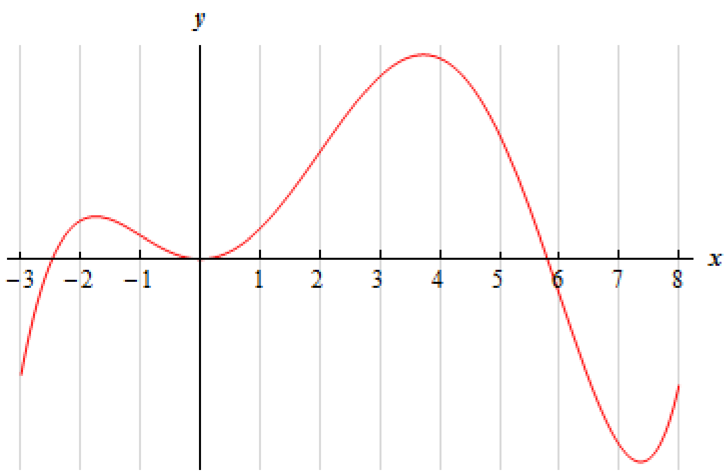
Preface

Here are the solutions to the practice problems for my Calculus I notes. Some solutions will have more or less detail than other solutions. The level of detail in each solution will depend up on several issues. If the section is a review section, this mostly applies to problems in the first chapter, there will probably not be as much detail to the solutions given that the problems really should be review. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you've reached the level of working the harder problems then you will probably already understand the basics fairly well and won't need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven't been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

The Shape of a Graph, Part II

1. The graph of a function is given below. Determine the open intervals on which the function is concave up and concave down.



Solution

There really isn't too much to this problem. We can easily see from the graph where the function in concave up/concave down and so all we need to do is estimate where the concavity changes (and this really will be an estimate as it won't always be clear) and write down the intervals.

$$\boxed{\text{Concave Up : } \left(-1,2\right) \;\&\; \left(6,\infty\right) \qquad \text{Concave Down : } \left(-\infty,-1\right) \;\&\; \left(2,6\right)}$$

Again, the endpoints of these intervals are, at best, estimates as it won't always be clear just where the concavity changes.

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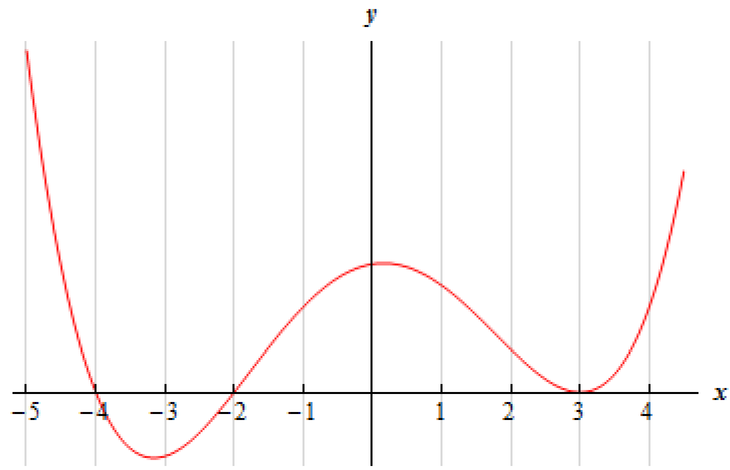
2. Below is the graph the **2nd derivative** of a function. From this graph determine the open intervals in which the **function** is concave up and concave down.



Hint : Be careful with this problem. The graph is of the **2nd derivative** of the function and so we don't just write down intervals where the graph is concave up and concave down. Recall how the 2nd derivative tells us where the function is concave up and concave down and this problem is not too bad.

Solution
We need to be careful and not do this problem as we did the first practice problem. The graph given is the graph of the **2nd derivative** and not the graph of the function. Therefore, the answer is not just where the graph is concave up or concave down.

What we need to do here is to recall that if the 2nd derivative is positive (*i.e.* the graph is above the *x*-axis) then the function in concave up and if the 2nd derivative is negative (*i.e.* the graph is below the *x*-axis) then the function is concave down.

So, it is fairly clear where the graph is above/below the *x*-axis and so we have the following intervals of concave up/concave down.

$$\boxed{\text{Concave Up : } (-\infty,-4), (-2,3) \ \& \ (3,\infty) \qquad \text{Concave Down : } (-4,-2)}$$

Even though the problem didn't ask for it we can also identify that $x = -4$ and $x = -2$ are inflection points because at these points the concavity changes. Note that $x = 3$ is **not** an inflection point. Clearly the 2nd derivative is zero here, but the concavity doesn't change at this point and so it is not an inflection point. Keep in mind inflection points are only where the concavity changes and not simply where the 2nd derivative is zero.

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3. For $f(x) = 12 + 6x^2 - x^3$ answer each of the following questions.
   (a) Determine a list of possible inflection points for the function.
   (b) Determine the open intervals on which the function is concave up and concave down.
   (c) Determine the inflection points of the function.

(a) Determine a list of possible inflection points for the function.

To get the list of possible inflection points for the function we'll need the 2nd derivative of the function so here that is.

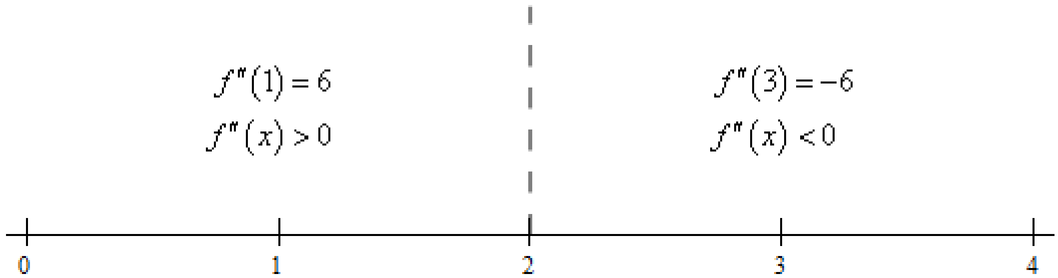
$$f'(x) = 12x - 3x^2 \qquad \underline{f''(x) = 12 - 6x}$$

Now, recall that possible inflection points are where the 2nd derivative either doesn't exist or is zero. Clearly the 2nd derivative exists everywhere (it's a polynomial….) and, in this case, it should be fairly clear where the 2nd derivative is zero. The only possible inflection critical point of the function in this case is,

$$\underline{x = 2}$$

(b) Determine the open intervals on which the function is concave up and concave down.

There isn't much to this part. All we really need here is a number line for the 2nd derivative. Here that is,



From this we get the following concave up/concave down information for the function.

$$\boxed{\text{Concave Up : } (-\infty, 2) \qquad\qquad \text{Concave Down : } (2, \infty)}$$

(c) Determine the inflection points of the function.

For this part all we need to do is interpret the results from the previous step. Recall that inflection points are points where the concavity changes (as opposed to simply the points where the 2nd derivative is zero or doesn't exist). Therefore the single inflection point for this function is,

$$\boxed{x = 2}$$

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4. For $g(z) = z^4 - 12z^3 + 84z + 4$ answer each of the following questions.

- **(a)** Determine a list of possible inflection points for the function.
- **(b)** Determine the open intervals on which the function is concave up and concave down.
- **(c)** Determine the inflection points of the function.

(a) Determine a list of possible inflection points for the function.

To get the list of possible inflection points for the function we'll need the 2nd derivative of the function so here that is.

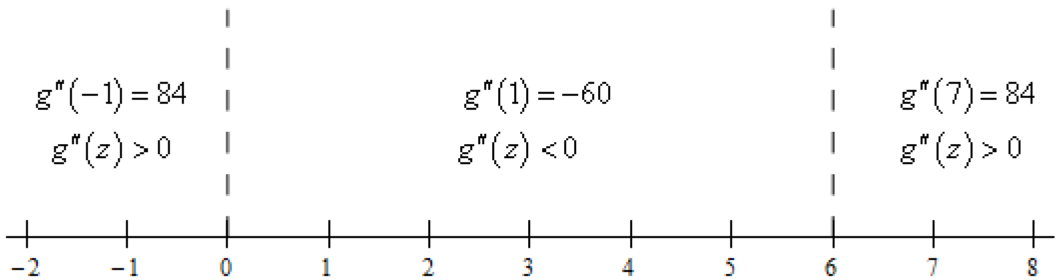
$$g'(z) = 4z^3 - 36z^2 + 84 \qquad\qquad \underline{g''(z) = 12z^2 - 72z = 12z(z-6)}$$

Now, recall that possible inflection points are where the 2nd derivative either doesn't exist or is zero. Clearly the 2nd derivative exists everywhere (it's a polynomial….) and, because we factored the 2nd derivative, it should be fairly clear where the 2nd derivative is zero. The possible inflection critical points of this function are,

$$\underline{z = 0 \quad \& \quad z = 6}$$

(b) Determine the open intervals on which the function is concave up and concave down.

There isn't much to this part. All we really need here is a number line for the 2nd derivative. Here that is,



From this we get the following concave up/concave down information for the function.

$$\boxed{\text{Concave Up}: \left(-\infty,0\right) \ \ \& \ \ \left(6,\infty\right) \qquad\qquad \text{Concave Down}: \ \left(0,6\right)}$$

(c) Determine the inflection points of the function.

For this part all we need to do is interpret the results from the previous step. Recall that inflection points are points where the concavity changes (as opposed to simply the points where the 2nd derivative is zero or doesn't exist). Therefore the inflection points for this function are,

$$\boxed{z=0 \quad \& \quad z=6}$$

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5. For $h\left(t\right)=t^4+12t^3+6t^2-36t+2$ answer each of the following questions.

   (a) Determine a list of possible inflection points for the function.
   (b) Determine the open intervals on which the function is concave up and concave down.
   (c) Determine the inflection points of the function.

(a) Determine a list of possible inflection points for the function.

To get the list of possible inflection points for the function we'll need the 2nd derivative of the function so here that is.

$$h'\left(t\right)=4t^3+36t^2+12t-36 \qquad\qquad h''\left(t\right)=12t^2+72t+12=12\left(t^2+6t+1\right)$$

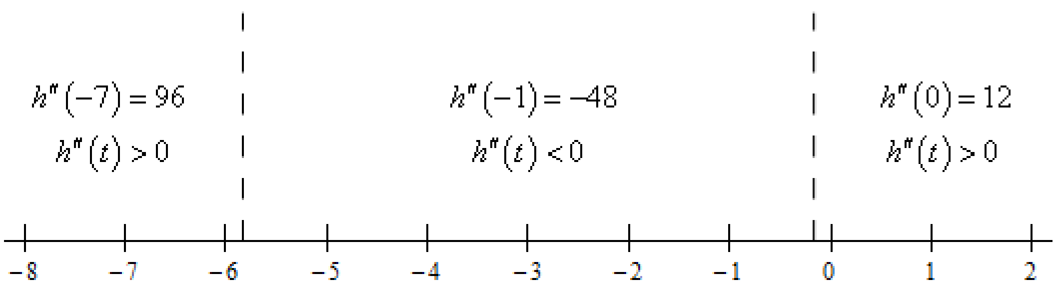
Now, recall that possible inflection points are where the 2nd derivative either doesn't exist or is zero. Clearly the 2nd derivative exists everywhere (it's a polynomial….). In this case the 2nd derivative doesn't factor and so we'll need to use the quadratic formula to determine where the 2nd derivative is zero.

The possible inflection critical points of this function are,

$$\underline{t = -3 \pm 2\sqrt{2} = -5.8284, \ -0.1716}$$

(b) Determine the open intervals on which the function is concave up and concave down.

There isn't much to this part. All we really need here is a number line for the 2nd derivative. Here that is,



From this we get the following concave up/concave down information for the function.

$$\boxed{\begin{aligned} \text{Concave Up}: & \left(-\infty, -3-2\sqrt{2}\right) \quad \& \quad \left(-3+2\sqrt{2}, \infty\right) \\ \text{Concave Down}: & \left(-3-2\sqrt{2}, -3+2\sqrt{2}\right) \end{aligned}}$$

(c) Determine the inflection points of the function.

For this part all we need to do is interpret the results from the previous step. Recall that inflection points are points where the concavity changes (as opposed to simply the points where the 2nd derivative is zero or doesn't exist). Therefore the inflection points for this function are,

$$\boxed{t = -3-2\sqrt{2} \quad \& \quad t = -3+2\sqrt{2}}$$

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6. For $h(w) = 8 - 5w + 2w^2 - \cos(3w)$ on $[-1, 2]$ answer each of the following questions.

(a) Determine a list of possible inflection points for the function.
(b) Determine the open intervals on which the function is concave up and concave down.

(c) Determine the inflection points of the function.

(a) Determine a list of possible inflection points for the function.

To get the list of possible inflection points for the function we'll need the 2nd derivative of the function so here that is.

$$h'(w) = -5 + 4w + 3\sin(3w) \qquad\qquad \underline{h''(w) = 4 + 9\cos(3w)}$$

Now, recall that possible inflection points are where the 2nd derivative either doesn't exist or is zero. Clearly the 2nd derivative exists everywhere (the cosine function exists everywhere…) and so all we need to do is set the 2nd derivative equal to zero and solve. We're not going to show all of those details so if you need to do some review of the process go back to the Solving Trig Equations sections for some examples.

The possible inflection critical points of this function are,

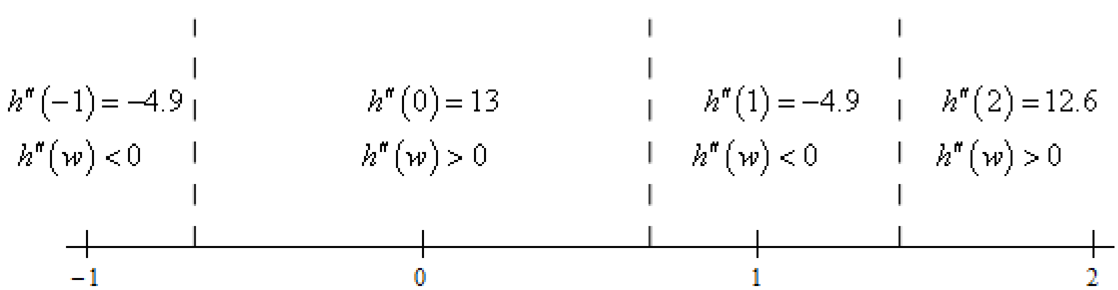
$$\begin{aligned} w &= 0.6771 + \tfrac{2\pi}{3}n \\ w &= 1.4173 + \tfrac{2\pi}{3}n \end{aligned} \qquad\qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Plugging in some n's gives the following possible inflection points in the interval $[-1, 2]$.

$$\underline{w = -0.6771 \qquad w = 0.6771 \qquad w = 1.4173}$$

(b) Determine the open intervals on which the function is concave up and concave down.

There isn't much to this part. All we really need here is a number line for the 2nd derivative. Here that is,



From this we get the following concave up/concave down information for the function.

$$\boxed{\begin{array}{c}\text{Concave Up}: \left(-0.6771,\ 0.6771\right) \quad \& \quad \left(1.4173,\ 2\right] \\ \text{Concave Down}: \left[-1,\ -0.6771\right) \quad \& \quad \left(0.6771,\ 1.4173\right)\end{array}}$$

Be careful with the end points of these intervals! We are working on the interval $\left[-1,2\right]$ and we've done no work for concavity outside of this interval and so we can't say anything about what happens outside of the interval.

(c) Determine the inflection points of the function.

For this part all we need to do is interpret the results from the previous step. Recall that inflection points are points where the concavity changes (as opposed to simply the points where the 2nd derivative is zero or doesn't exist). Therefore the inflection points for this function are,

$$\boxed{w=-0.6771 \qquad w=0.6771 \qquad w=1.4173}$$

As with the previous step we have to be careful and recall that we are working on the interval $\left[-1,2\right]$. There are infinitely many more possible inflection points and we've done no work outside of the interval to determine if they are in fact inflection points!

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7. For $R\left(z\right)=z\left(z+4\right)^{\frac{2}{3}}$ answer each of the following questions.

(a) Determine a list of possible inflection points for the function.
(b) Determine the open intervals on which the function is concave up and concave down.
(c) Determine the inflection points of the function.

(a) Determine a list of possible inflection points for the function.

To get the list of possible inflection points for the function we'll need the 2nd derivative of the function so here that is.

$$R'\left(z\right)=\left(z+4\right)^{\frac{2}{3}}+z\left(\tfrac{2}{3}\right)\left(z+4\right)^{-\frac{1}{3}}=\frac{5z+12}{3\left(z+4\right)^{\frac{1}{3}}}$$

$$R''\left(z\right)=\frac{5\left(3\left(z+4\right)^{\frac{1}{3}}\right)-\left(5z+12\right)\left(z+4\right)^{-\frac{2}{3}}}{\left[3\left(z+4\right)^{\frac{1}{3}}\right]^{2}}=\frac{\left[15\left(z+4\right)-\left(5z+12\right)\right]\left(z+4\right)^{-\frac{2}{3}}}{9\left(z+4\right)^{\frac{2}{3}}}=\underline{\frac{10z+48}{9\left(z+4\right)^{\frac{4}{3}}}}$$

Note that we simplified the derivatives at each step to help with the next step. You don't technically need to do this, but having the 2nd derivative in its "simplest" form will definitely help with getting the answer to this part.

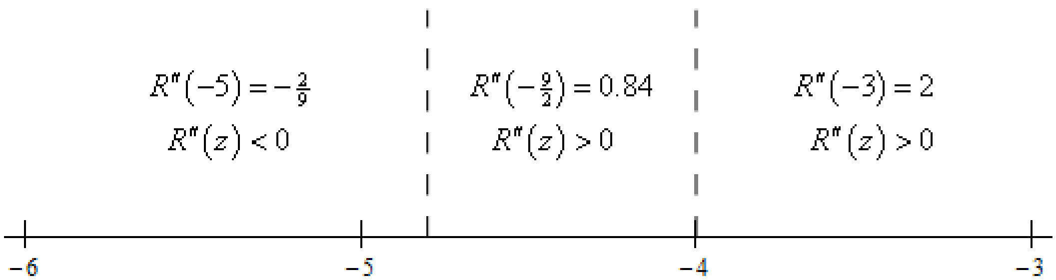
Now, recall that possible inflection points are where the 2nd derivative either doesn't exist or is zero. Because we simplified the 2nd derivative as much as possible it is clear that the 2nd derivative won't exist at $z = -4$ (and the function exists at this point as well!). It should also be clear that the 2nd derivative is zero at $z = -\frac{48}{10} = -\frac{24}{5}$.

The possible inflection critical points of this function are then,

$$\underline{z = -\tfrac{48}{5} = -4.8 \qquad \& \qquad z = -4}$$

(b) Determine the open intervals on which the function is concave up and concave down.

There isn't much to this part. All we really need here is a number line for the 2nd derivative. Here that is,



From this we get the following concave up/concave down information for the function.

$$\boxed{\text{Concave Up : } \left(-\tfrac{24}{5}, -4\right) \quad \& \quad \left(-4, \infty\right) \qquad \text{Concave Down : } \left(-\infty, -\tfrac{24}{5}\right)}$$

(c) Determine the inflection points of the function.

For this part all we need to do is interpret the results from the previous step. Recall that inflection points are points where the concavity changes (as opposed to simply the points where the 2nd derivative is zero or doesn't exist). Therefore the only inflection point for this function is,

$$\boxed{z = -\tfrac{24}{5}}$$

8. For $h(x) = \mathbf{e}^{4-x^2}$ answer each of the following questions.

(a) Determine a list of possible inflection points for the function.
(b) Determine the open intervals on which the function is concave up and concave down.
(c) Determine the inflection points of the function.

(a) Determine a list of possible inflection points for the function.

To get the list of possible inflection points for the function we'll need the 2nd derivative of the function so here that is.

$$h'(x) = -2x\mathbf{e}^{4-x^2} \qquad\qquad h''(x) = -2\mathbf{e}^{4-x^2} + 4x^2\mathbf{e}^{4-x^2} = \underline{2\mathbf{e}^{4-x^2}\left(2x^2-1\right)}$$

Don't forget to product rule for the 2nd derivative and factoring the exponential out will help a little with the next step.

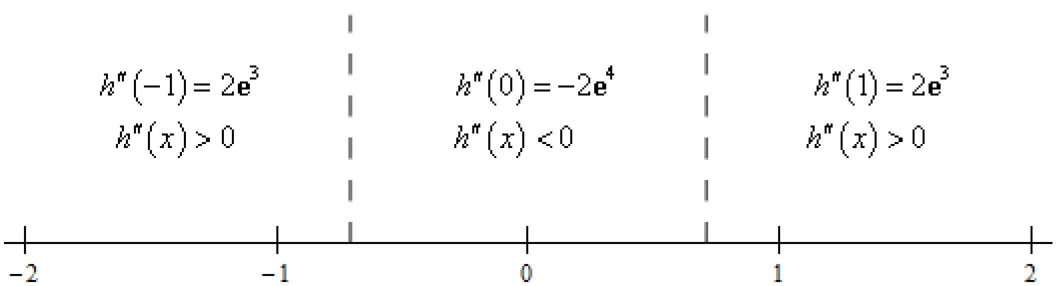
Now, recall that possible inflection points are where the 2nd derivative either doesn't exist or is zero. It should be fairly clear that the 2nd derivative exists everywhere (it is a product of two functions that exist everywhere…). We also know that exponentials are never zero and so the 2nd derivative will be zero at the solutions to $2x^2 - 1 = 0$

The possible inflection critical points of this function are then,

$$\underline{x = \pm\tfrac{1}{\sqrt{2}} = \pm 0.7071}$$

(b) Determine the open intervals on which the function is concave up and concave down.

There isn't much to this part. All we really need here is a number line for the 2nd derivative. Here that is,



From this we get the following concave up/concave down information for the function.

$$\boxed{\text{Concave Up : } \left(-\infty, -\tfrac{1}{\sqrt{2}}\right) \ \& \ \left(\tfrac{1}{\sqrt{2}}, \infty\right) \qquad \text{Concave Down : } \left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right)}$$

(c) Determine the inflection points of the function.

For this part all we need to do is interpret the results from the previous step. Recall that inflection points are points where the concavity changes (as opposed to simply the points where the 2nd derivative is zero or doesn't exist). Therefore the only inflection point for this function is,

$$\boxed{x = -\tfrac{1}{\sqrt{2}} = -0.7071 \qquad x = \tfrac{1}{\sqrt{2}} = 0.7071}$$

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9. For $g(t) = t^5 - 5t^4 + 8$ answer each of the following questions.

(a) Identify the critical points of the function.
(b) Determine the open intervals on which the function increases and decreases.
(c) Classify the critical points as relative maximums, relative minimums or neither.
(d) Determine the open intervals on which the function is concave up and concave down.
(e) Determine the inflection points of the function.
(f) Use the information from steps **(a)** – **(e)** to sketch the graph of the function.

(a) Identify the critical points of the function.

The parts to this problem (with the exception of the last part) are just like the basic increasing/decreasing problems from the previous section and the basic concavity problems from earlier in this section. Because of that we will not be putting in quite as much detail here. If you are still unsure how to work the parts of this problem you should go back and work a few of the basic problems from the previous section and/or earlier in this section before proceeding.

We will need the 1st derivative to start things off.

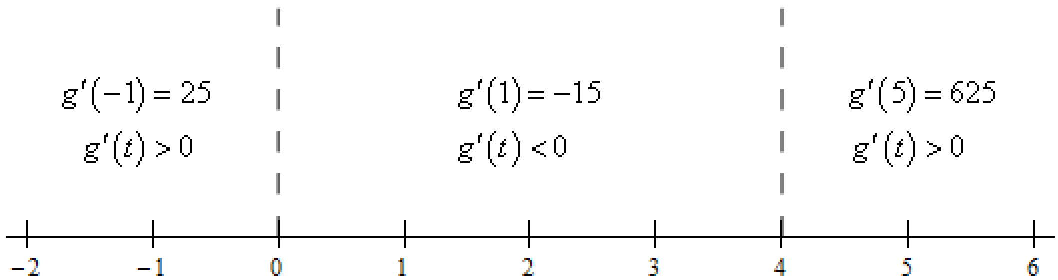
$$\underline{g'(t) = 5t^4 - 20t^3 = 5t^3(t-4)}$$

From the 1st derivative we can see that the critical points of this function are then,

$$\underline{t = 0 \quad \& \quad t = 4}$$

(b) Determine the open intervals on which the function increases and decreases.

To answer this part all we need is the number line for the 1st derivative.



From this we get the following increasing/decreasing information for the function.

$$\boxed{\text{Increasing : } (-\infty,0) \ \ \& \ \ (4,\infty) \qquad\qquad \text{Decreasing : } \ (0,4)}$$

(c) Classify the critical points as relative maximums, relative minimums or neither.

From the number line in the previous step we get the following classifications of the critical points.

$$\boxed{t=0: \text{ Relative Maximum} \qquad\qquad t=4: \text{ Relative Minimum}}$$

(d) Determine the open intervals on which the function is concave up and concave down.

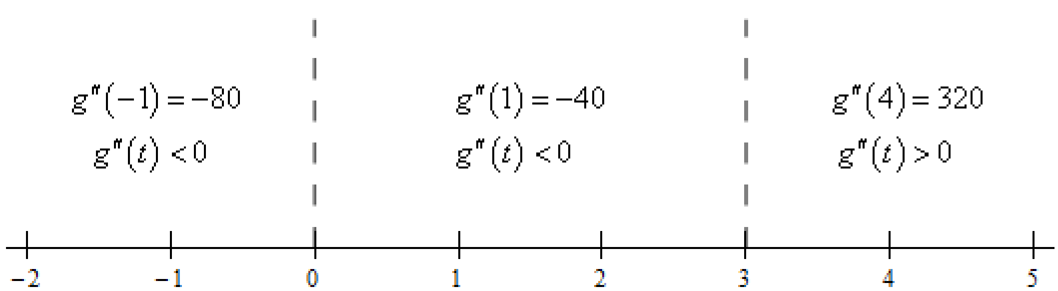
We'll need the 2nd derivative to find the list of possible inflection points.

$$\underline{g''(t)=20t^3-60t^2=20t^2(t-3)}$$

The possible inflection points for this problem are,

$$\underline{t=0 \qquad \& \qquad t=3}$$

To get the intervals of concavity we'll need the number line for the 2nd derivative.



From this we get the following concavity information for the function.

$$\boxed{\text{Concave Up}: \left(3,\infty\right) \qquad \text{Concave Down}: \left(-\infty,0\right) \quad \& \quad \left(0,3\right)}$$

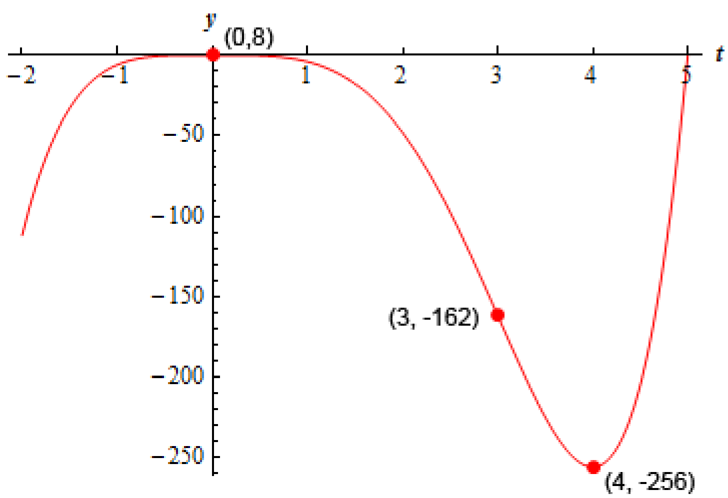
(e) Determine the inflection points of the function.

From the concavity information in the previous step we can see that the single inflection point for the function is,

$$\boxed{t = 3}$$

(f) Use the information from steps **(a)** – **(e)** to sketch the graph of the function.

Here is a sketch of the graph of this function using the information above. As we did in problems in this section we can start at the left and work our way to the right on the graph. As we do this we first pay attention to the increasing/decreasing information and then make sure that the curve has the correct concavity as we sketch it in.



Note that because we used a computer to generate the sketch it is possible that your sketch won't be quite the same. It should however, have the same points listed on the graph above, the same basic increasing/decreasing nature and the same basic concavity.

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10. For $f\left(x\right) = 5 - 8x^3 - x^4$ answer each of the following questions.

(a) Identify the critical points of the function.
(b) Determine the open intervals on which the function increases and decreases.
(c) Classify the critical points as relative maximums, relative minimums or neither.
(d) Determine the open intervals on which the function is concave up and concave down.

(e) Determine the inflection points of the function.
(f) Use the information from steps **(a)** – **(e)** to sketch the graph of the function.

(a) Identify the critical points of the function.

The parts to this problem (with the exception of the last part) are just like the basic increasing/decreasing problems from the previous section and the basic concavity problems from earlier in this section. Because of that we will not be putting in quite as much detail here. If you are still unsure how to work the parts of this problem you should go back and work a few of the basic problems from the previous section and/or earlier in this section before proceeding.

We will need the 1st derivative to start things off.

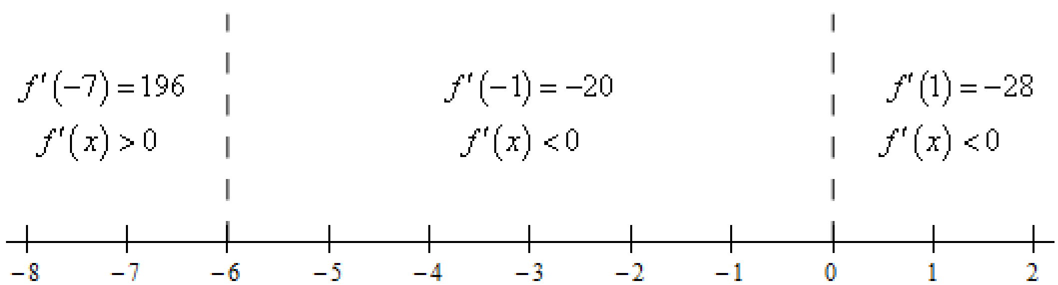
$$\underline{f'(x) = -24x^2 - 4x^3 = -4x^2(x+6)}$$

From the 1st derivative we can see that the critical points of this function are then,

$$\underline{x = -6 \quad \& \quad x = 0}$$

(b) Determine the open intervals on which the function increases and decreases.

To answer this part all we need is the number line for the 1st derivative.



From this we get the following increasing/decreasing information for the function.

$$\boxed{\text{Increasing : } (-\infty,-6) \qquad\qquad \text{Decreasing : } (-6,0) \;\;\&\;\; (0,\infty)}$$

(c) Classify the critical points as relative maximums, relative minimums or neither.

From the number line in the previous step we get the following classifications of the critical points.

$$\boxed{x=-6:\ \text{Relative Maximum} \qquad\qquad x=0:\ \text{Neither}}$$

(d) Determine the open intervals on which the function is concave up and concave down.

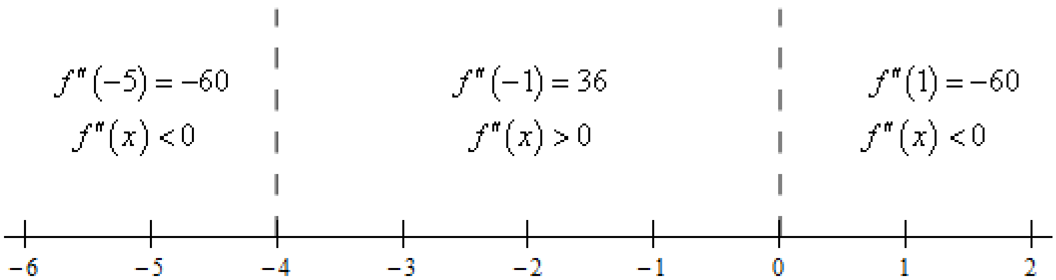
We'll need the 2nd derivative to find the list of possible inflection points.

$$f''(x) = -48x - 12x^2 = -12x(x+4)$$

The possible inflection points for this function are,

$$x=-4 \qquad \& \qquad x=0$$

To get the intervals of concavity we'll need the number line for the 2nd derivative.



From this we get the following concavity information for the function.

$$\boxed{\text{Concave Up}:\ (-4,0) \qquad\qquad \text{Concave Down}:\ (-\infty,-4) \quad \& \quad (0,\infty)}$$

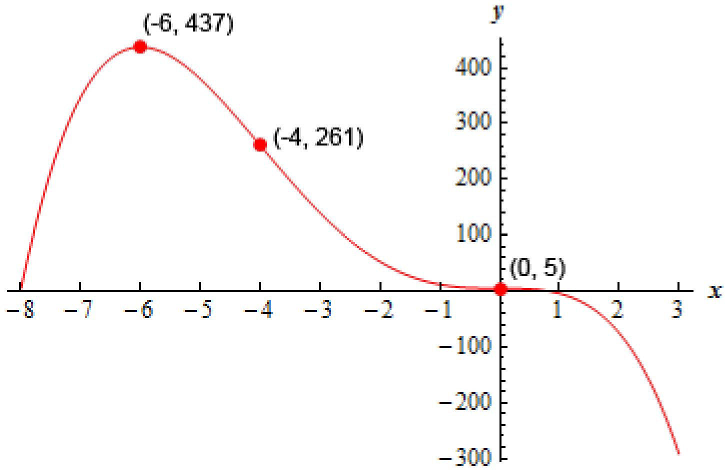
(e) Determine the inflection points of the function.

From the concavity information in the previous step we can see that the inflection points for the function are,

$$\boxed{x=-4 \qquad \& \qquad x=0}$$

(f) Use the information from steps **(a)** – **(e)** to sketch the graph of the function.

Here is a sketch of the graph of this function using the information above. As we did in problems in this section we can start at the left and work our way to the right on the graph. As we do this we first pay attention to the increasing/decreasing information and then make sure that the curve has the correct concavity as we sketch it in.



Note that because we used a computer to generate the sketch it is possible that your sketch won't be quite the same. It should however, have the same points listed on the graph above, the same basic increasing/decreasing nature and the same basic concavity.

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11. For $h(z) = z^4 - 2z^3 - 12z^2$ answer each of the following questions.

(a) Identify the critical points of the function.
(b) Determine the open intervals on which the function increases and decreases.
(c) Classify the critical points as relative maximums, relative minimums or neither.
(d) Determine the open intervals on which the function is concave up and concave down.
(e) Determine the inflection points of the function.
(f) Use the information from steps **(a)** – **(e)** to sketch the graph of the function.

(a) Identify the critical points of the function.

The parts to this problem (with the exception of the last part) are just like the basic increasing/decreasing problems from the previous section and the basic concavity problems from earlier in this section. Because of that we will not be putting in quite as much detail here. If you are still unsure how to work the parts of this problem you should go back and work a few of the basic problems from the previous section and/or earlier in this section before proceeding.

We will need the 1st derivative to start things off.

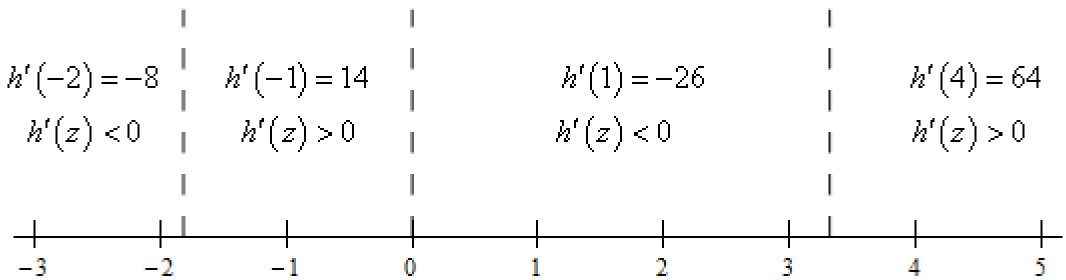
$$\underline{h'(z) = 4z^3 - 6z^2 - 24z = 2z(2z^2 - 3z - 12)}$$

From the 1st derivative we can see that the critical points of this function are then,

$$\underline{x = 0 \qquad \& \qquad x = \tfrac{3 \pm \sqrt{105}}{4} = -1.8117,\ 3.3117}$$

(b) Determine the open intervals on which the function increases and decreases.

To answer this part all we need is the number line for the 1st derivative.



From this we get the following increasing/decreasing information for the function.

$$\boxed{\text{Increasing : } \left(\tfrac{3-\sqrt{105}}{4}, 0\right) \ \ \& \ \ \left(\tfrac{3+\sqrt{105}}{4}, \infty\right) \qquad \text{Decreasing : } \left(-\infty, \tfrac{3-\sqrt{105}}{4}\right) \ \ \& \ \ \left(0, \tfrac{3+\sqrt{105}}{4}\right)}$$

(c) Classify the critical points as relative maximums, relative minimums or neither.

From the number line in the previous step we get the following classifications of the critical points.

$$\boxed{\begin{array}{c} z = \tfrac{3-\sqrt{105}}{4} : \text{ Relative Minimum} \qquad z = 0 : \text{ Relative Maximum} \\ z = \tfrac{3+\sqrt{105}}{4} : \text{ Relative Minimum} \end{array}}$$

(d) Determine the open intervals on which the function is concave up and concave down.

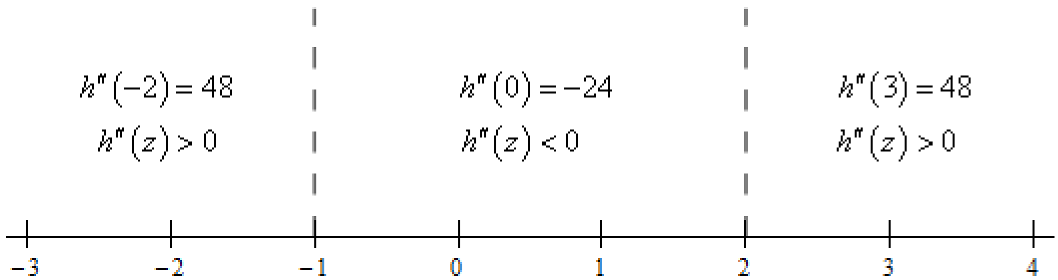
We'll need the 2nd derivative to find the list of possible inflection points.

$$\underline{h''(z) = 12z^2 - 12z - 24 = 12(z-2)(z+1)}$$

The possible inflection points for this function are,

$$\underline{z = -1 \qquad \& \qquad z = 2}$$

To get the intervals of concavity we'll need the number line for the 2nd derivative.



From this we get the following concavity information for the function.

$$\boxed{\text{Concave Up : } (-\infty,-1) \quad \& \quad (2,\infty) \qquad\qquad \text{Concave Down : } (-1,2)}$$

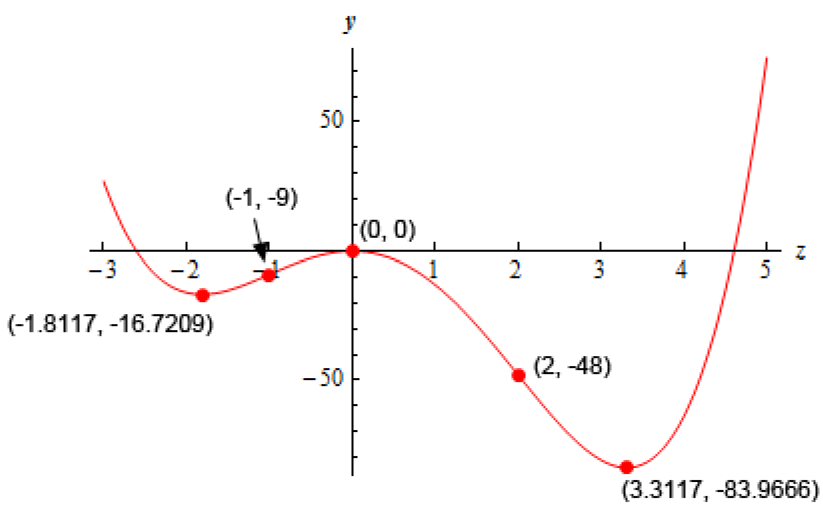
(e) Determine the inflection points of the function.

From the concavity information in the previous step we can see that the inflection points for the function are,

$$\boxed{z=-1 \qquad \& \qquad z=2}$$

(f) Use the information from steps **(a)** – **(e)** to sketch the graph of the function.

Here is a sketch of the graph of this function using the information above. As we did in problems in this section we can start at the left and work our way to the right on the graph. As we do this we first pay attention to the increasing/decreasing information and then make sure that the curve has the correct concavity as we sketch it in.



Note that because we used a computer to generate the sketch it is possible that your sketch won't be quite the same. It should however, have the same points listed on the graph above, the same basic increasing/decreasing nature and the same basic concavity.

12. For $Q(t) = 3t - 8\sin\left(\frac{t}{2}\right)$ on $[-7,4]$ answer each of the following questions.

(a) Identify the critical points of the function.
(b) Determine the open intervals on which the function increases and decreases.
(c) Classify the critical points as relative maximums, relative minimums or neither.
(d) Determine the open intervals on which the function is concave up and concave down.
(e) Determine the inflection points of the function.
(f) Use the information from steps **(a)** – **(e)** to sketch the graph of the function.

(a) Identify the critical points of the function.

The parts to this problem (with the exception of the last part) are just like the basic increasing/decreasing problems from the previous section and the basic concavity problems from earlier in this section. Because of that we will not be putting in quite as much detail here. If you are still unsure how to work the parts of this problem you should go back and work a few of the basic problems from the previous section and/or earlier in this section before proceeding.

We will need the 1st derivative to start things off.

$$\underline{Q'(t) = 3 - 4\cos\left(\tfrac{t}{2}\right)}$$

From the 1st derivative all of the critical points are,

$$\begin{aligned} t &= 1.4454 + 4\pi n \\ t &= 11.1210 + 4\pi n \end{aligned} \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

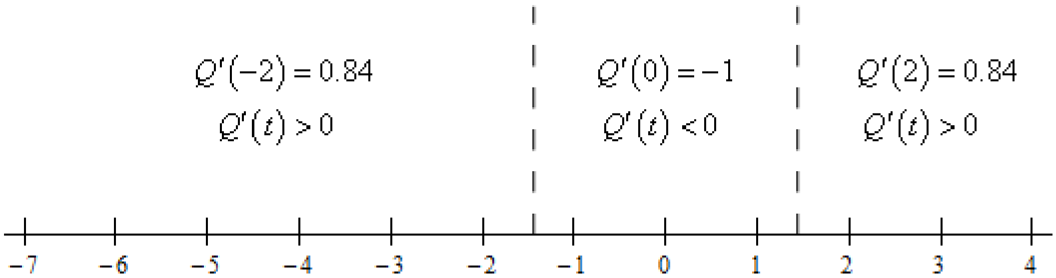
If you need some review of the solving trig equation process go back to the Solving Trig Equations sections for some examples.

Plugging in some values of n we see that the critical points in the interval $[-7,4]$ are,

$$\underline{t = -1.4454 \qquad \& \qquad t = 1.4454}$$

(b) Determine the open intervals on which the function increases and decreases.

To answer this part all we need is the number line for the 1st derivative.



From this we get the following increasing/decreasing information for the function.

$$\boxed{\text{Increasing : } [-7,-1.4454) \quad \& \quad (1.4454,4] \qquad\qquad \text{Decreasing : } (-1.4454,1.4454)}$$

(c) Classify the critical points as relative maximums, relative minimums or neither.

From the number line in the previous step we get the following classifications of the critical points.

$$\boxed{t=-1.4454: \text{ Relative Maximum} \qquad\qquad t=1.4454: \text{ Relative Minimum}}$$

(d) Determine the open intervals on which the function is concave up and concave down.

We'll need the 2nd derivative to find the list of possible inflection points.

$$\underline{Q''(t) = 2\sin\left(\tfrac{t}{2}\right)}$$

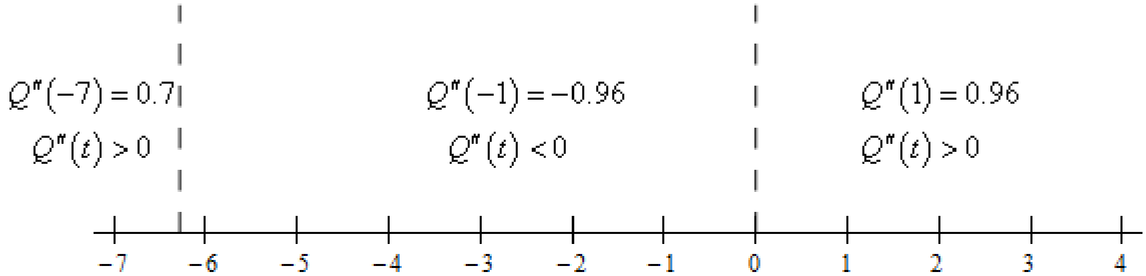
All possible critical points of the function are,

$$\begin{aligned} t &= 4\pi n \\ t &= 2\pi + 4\pi n \end{aligned} \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Plugging in some values of n we see that the possible inflection points in the interval $[-7,4]$ are,

$$\underline{t = -6.2832 \qquad \& \qquad t = 0}$$

To get the intervals of concavity we'll need the number line for the 2nd derivative.



From this we get the following concavity information for the function.

$$\boxed{\text{Concave Up}: \left[-7,-6.2832\right) \quad \& \quad \left(0,4\right] \qquad\qquad \text{Concave Down}: \left(-6.2832,0\right)}$$

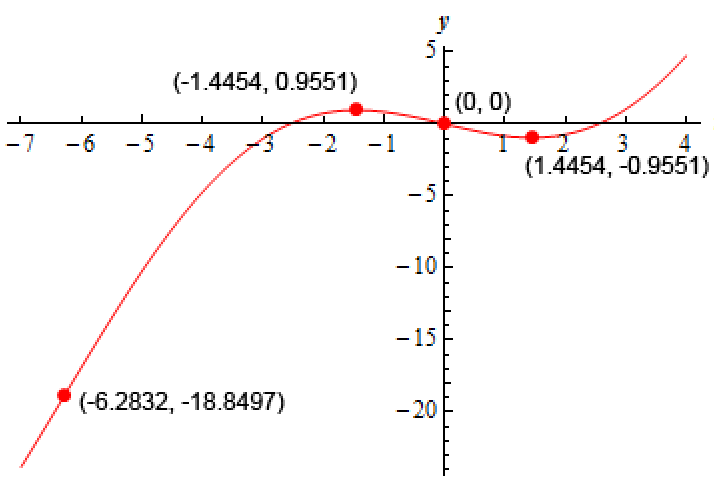
(e) Determine the inflection points of the function.

From the concavity information in the previous step we can see that the inflection points for the function are,

$$\boxed{t=-6.2832 \qquad \& \qquad t=0}$$

(f) Use the information from steps **(a)** – **(e)** to sketch the graph of the function.

Here is a sketch of the graph of this function using the information above. As we did in problems in this section we can start at the left and work our way to the right on the graph. As we do this we first pay attention to the increasing/decreasing information and then make sure that the curve has the correct concavity as we sketch it in.



Note that because we used a computer to generate the sketch it is possible that your sketch won't be quite the same. It should however, have the same points listed on the graph above, the same basic increasing/decreasing nature and the same basic concavity.

13. For $f(x) = x^{\frac{4}{3}}(x-2)$ answer each of the following questions.

(a) Identify the critical points of the function.
(b) Determine the open intervals on which the function increases and decreases.
(c) Classify the critical points as relative maximums, relative minimums or neither.
(d) Determine the open intervals on which the function is concave up and concave down.
(e) Determine the inflection points of the function.
(f) Use the information from steps **(a)** – **(e)** to sketch the graph of the function.

(a) Identify the critical points of the function.

The parts to this problem (with the exception of the last part) are just like the basic increasing/decreasing problems from the previous section and the basic concavity problems from earlier in this section. Because of that we will not be putting in quite as much detail here. If you are still unsure how to work the parts of this problem you should go back and work a few of the basic problems from the previous section and/or earlier in this section before proceeding.

We will need the 1st derivative to start things off.

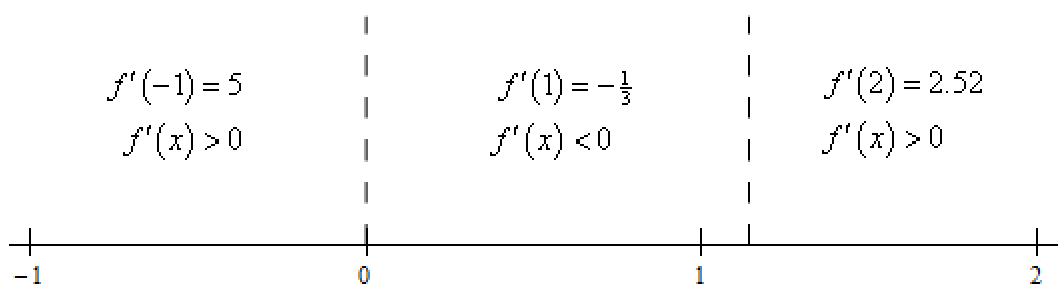
$$f(x) = x^{\frac{7}{3}} - 2x^{\frac{4}{3}} \qquad \to \qquad \underline{f'(x) = \tfrac{7}{3}x^{\frac{4}{3}} - \tfrac{8}{3}x^{\frac{1}{3}} = \tfrac{1}{3}x^{\frac{1}{3}}(7x-8)}$$

Note that by factoring the $x^{\frac{1}{3}}$ out we made it a little easier to quickly see that the critical points are,

$$\underline{x = 0 \quad \& \quad x = \tfrac{8}{7} = 1.1429}$$

(b) Determine the open intervals on which the function increases and decreases.

To answer this part all we need is the number line for the 1st derivative.



From this we get the following increasing/decreasing information for the function.

$$\boxed{\text{Increasing : } (-\infty,0) \quad \& \quad \left(\tfrac{8}{7},\infty\right) \qquad \text{Decreasing : } \left(0,\tfrac{8}{7}\right)}$$

(c) Classify the critical points as relative maximums, relative minimums or neither.

From the number line in the previous step we get the following classifications of the critical points.

$$\boxed{x=0: \text{ Relative Maximum} \qquad\qquad x=\tfrac{8}{7}: \text{ Relative Minimum}}$$

(d) Determine the open intervals on which the function is concave up and concave down.

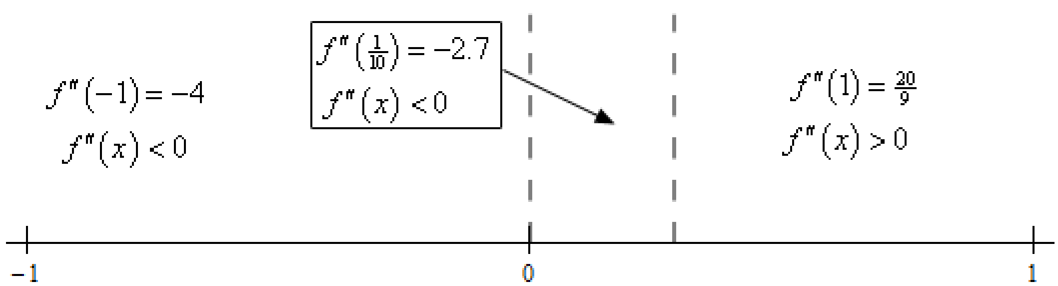
We'll need the 2nd derivative to find the list of possible inflection points.

$$\underline{f''(x) = \tfrac{28}{9}x^{\frac{1}{3}} - \tfrac{8}{9}x^{-\frac{2}{3}} = \frac{28x-8}{9x^{\frac{2}{3}}}}$$

The possible inflection points for this function are,

$$\underline{x=0 \qquad \& \qquad x=\tfrac{2}{7}=0.2857}$$

To get the intervals of concavity we'll need the number line for the 2nd derivative.



From this we get the following concavity information for the function.

$$\boxed{\text{Concave Up : } \left(\tfrac{2}{7},\infty\right) \qquad \text{Concave Down : } (-\infty,0) \quad \& \quad \left(0,\tfrac{2}{7}\right)}$$

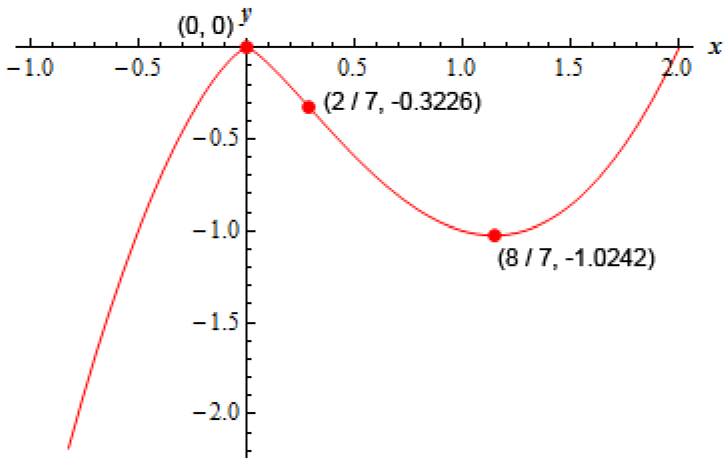
(e) Determine the inflection points of the function.

From the concavity information in the previous step we can see that the single inflection point for the function is,

$$\boxed{x = \tfrac{2}{7}}$$

(f) Use the information from steps **(a)** – **(e)** to sketch the graph of the function.

Here is a sketch of the graph of this function using the information above. As we did in problems in this section we can start at the left and work our way to the right on the graph. As we do this we first pay attention to the increasing/decreasing information and then make sure that the curve has the correct concavity as we sketch it in.



Note that because we used a computer to generate the sketch it is possible that your sketch won't be quite the same. It should however, have the same points listed on the graph above, the same basic increasing/decreasing nature and the same basic concavity.

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14. For $P(w) = w\mathbf{e}^{4w}$ on $\left[-2, \frac{1}{4}\right]$ answer each of the following questions.

(a) Identify the critical points of the function.
(b) Determine the open intervals on which the function increases and decreases.
(c) Classify the critical points as relative maximums, relative minimums or neither.
(d) Determine the open intervals on which the function is concave up and concave down.
(e) Determine the inflection points of the function.
(f) Use the information from steps **(a)** – **(e)** to sketch the graph of the function.

(a) Identify the critical points of the function.

The parts to this problem (with the exception of the last part) are just like the basic increasing/decreasing problems from the previous section and the basic concavity problems from earlier in this section. Because of that we will not be putting in quite as much detail here. If you are still unsure how to work the parts of this problem you should go back and work a few of the basic problems from the previous section and/or earlier in this section before proceeding.

Also note that the interval here is only because we haven't discussed L'Hospital's Rule yet (that comes in a few sections…) and that makes the behavior of the graph as $w \to \pm\infty$ a little trickier. Once we cover that section you might want to come back and eliminate the interval and see what the graph is.

We will need the 1st derivative to start things off.

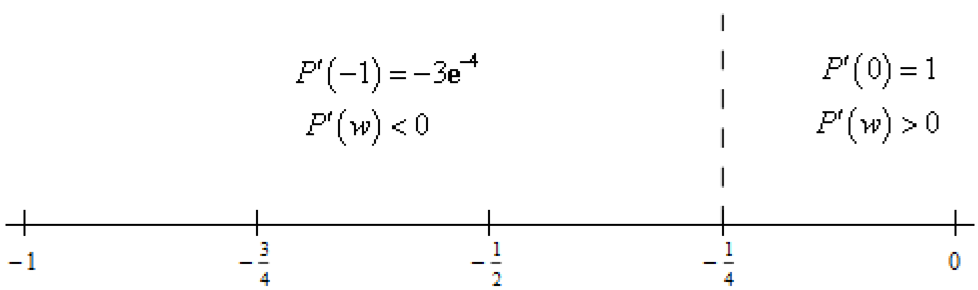
$$\underline{P'(w) = \mathbf{e}^{4w} + 4w\mathbf{e}^{4w} = \mathbf{e}^{4w}(1+4w)}$$

From the 1st derivative we can see that the only critical points of this function is,

$$\underline{w = -\tfrac{1}{4}}$$

(b) Determine the open intervals on which the function increases and decreases.

To answer this part all we need is the number line for the 1st derivative.



From this we get the following increasing/decreasing information for the function.

$$\boxed{\text{Increasing : } \left(-\tfrac{1}{4},\infty\right) \quad \text{Decreasing : } \left(-\infty,-\tfrac{1}{4}\right)}$$

(c) Classify the critical points as relative maximums, relative minimums or neither.

From the number line in the previous step we get the following classification of the critical point.

$$\boxed{w = -\tfrac{1}{4} : \text{ Relative Minimum}}$$

(d) Determine the open intervals on which the function is concave up and concave down.

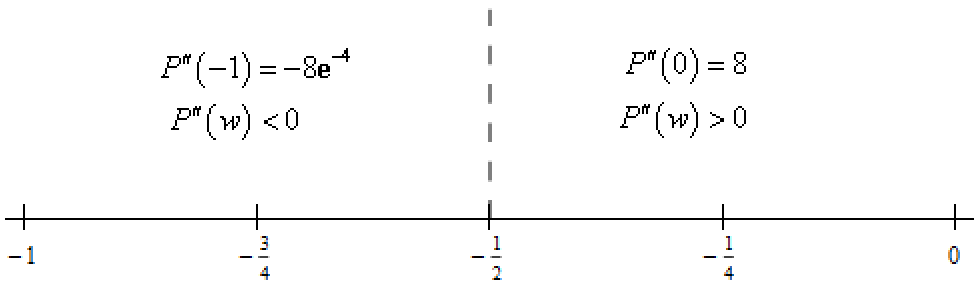
We'll need the 2nd derivative to find the list of possible inflection points.

$$\underline{P''(w) = \mathbf{e}^{4w}(4) + 4\mathbf{e}^{4w}(1+4w) = 8\mathbf{e}^{4w}(1+2w)}$$

The only possible inflection point for this function is,

$$\underline{w = -\tfrac{1}{2}}$$

To get the intervals of concavity we'll need the number line for the 2nd derivative.



From this we get the following concavity information for the function.

$$\boxed{\text{Concave Up : } \left(-\tfrac{1}{2}, \infty\right) \qquad \text{Concave Down : } \left(-\infty, -\tfrac{1}{2}\right)}$$

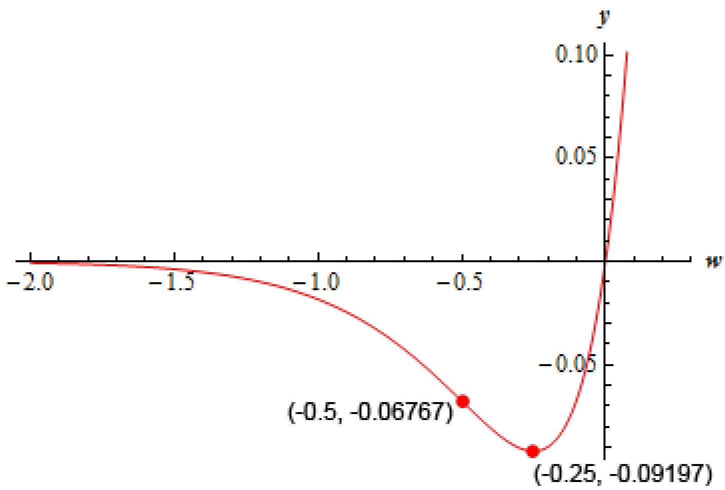
(e) Determine the inflection points of the function.

From the concavity information in the previous step we can see that the single inflection point for the function is,

$$\boxed{w = -\tfrac{1}{2}}$$

(f) Use the information from steps **(a)** – **(e)** to sketch the graph of the function.

Here is a sketch of the graph of this function using the information above. As we did in problems in this section we can start at the left and work our way to the right on the graph. As we do this we first pay attention to the increasing/decreasing information and then make sure that the curve has the correct concavity as we sketch it in.



Note that because we used a computer to generate the sketch it is possible that your sketch won't be quite the same. It should however, have the same points listed on the graph above, the same basic increasing/decreasing nature and the same basic concavity.

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15. Determine the minimum degree of a polynomial that has exactly one inflection point.

Hint : What is the simplest possible form of the 2nd derivative that we can have that will guarantee that we have a single inflection point?

Step 1

First, let's suppose that the single inflection point occurs at $x = a$ for some number *a*. The value of *a* is not important, this only allows us to discuss the problem.

Now, if we start with a polynomial, call it $p(x)$, then the 2nd derivative must also be a polynomial and we have to have $p''(a) = 0$. In addition we know that the 2nd derivative must change signs at $x = a$.

The simplest polynomial that we can have that will do this is,

$$\underline{p''(x) = x - a}$$

This clearly has $p''(a)=0$ and it will change sign at $x=a$. Note as well that we don't really care which side is concave up and which side is concave down. We only care that the 2nd derivative changes sign at $x=a$ as it does here.

Hint : We saw how to "undo" differentiation in the practice problems in the previous section. Here we simply need to do that twice and note that we don't actually have to undo the derivatives here, just think about what they would have to look like.

Step 2

Okay, saw how to "undo" differentiation in the practice problems of the previous section. We don't actually need to do that here, but we do need to think about what undoing differentiation will give here.

The 2nd derivative is a 1st degree polynomial and that means the 1st derivative had to be a 2nd degree polynomial. This should make sense to you if you understand how differentiation works.

We know that we have to differentiate the 1st derivative to get the 2nd derivative. Therefore because the highest power of *x* in the 2nd derivative is 1 and we know that differentiation lowers the power by 1 the highest power of *x* in the 1st derivative must have been 2.

Okay, we've figured out that the 1st derivative must have been a 2nd degree polynomial. This in turn means that the original function must have been a 3rd degree polynomial Again, differentiation lowers the power of *x* by 1 and if the highest power of *x* in the 1st derivative is 2 then the highest power of *x* in the original function must have been 3.

So, the minimum degree of a polynomial that has exactly one inflection point must be **three** (*i.e.* a cubic polynomial).

Note that we can have higher degree polynomials with exactly one inflection point. This is simply the minimal degree that will give exactly one inflection point.

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16. Suppose that we know that $f(x)$ is a polynomial with critical points $x=-1$, $x=2$ and $x=6$. If we also know that the 2nd derivative is $f''(x)=-3x^2+14x-4$. If possible, classify each of the critical points as relative minimums, relative maximums. If it is not possible to classify the critical points clearly explain why they cannot be classified.

Hint : We do **NOT** need the 1st derivative to answer this question. We are in the 2nd derivative section and we did see a way in the notes on how to use the 2nd derivative (which we have nicely been given…) to classify most critical points.

Solution

This problem is not as difficult as many students originally make it out to be. We've been given the 2nd derivative and we saw how the 2nd derivative test can be used to classify most critical points so let's use that.

First, we should note that because we have been told that $f(x)$ is a polynomial it should be fairly clear that, regardless of what the 1st derivative actually is, we should have,

$$f'(-1) = 0 \qquad\qquad f'(2) = 0 \qquad\qquad f'(6) = 0$$

What this means is that we can use the 2nd derivative test as it only works for these kinds of critical points.

All we need to do then is plug the critical points into the 2nd derivative and use the 2nd derivative test to classify the critical points.

$f''(-1) = -21 < 0$	Relative Maximum
$f''(2) = 12 > 0$	Relative Minimum
$f''(6) = -28 < 0$	Relative Maximum

So, in this case it was possible to classify all of the given critical points. Recall that if the 2nd derivative had been zero for any of them we would not have been able to classify that critical point without the 1st derivative which we don't have for this case.