Preface

Here are the solutions to the practice problems for my Calculus I notes. Some solutions will have more or less detail than other solutions. The level of detail in each solution will depend up on several issues. If the section is a review section, this mostly applies to problems in the first chapter, there will probably not be as much detail to the solutions given that the problems really should be review. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you've reached the level of working the harder problems then you will probably already understand the basics fairly well and won't need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven't been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

Optimization

1. Find two positive numbers whose sum is 300 and whose product is a maximum.

Step 1
The first step is to write down equations describing this situation.

Let's call the two numbers *x* and *y* and we are told that the sum is 300 (this is the constraint for the problem) or,

$$x + y = 300$$

We are being asked to maximize the product,

$$A = xy$$

Step 2
We now need to solve the constraint for *x* or *y* (and it really doesn't matter which variable we solve for in this case) and plug this into the product equation.

$$y = 300 - x \qquad \Rightarrow \qquad A(x) = x(300 - x) = 300x - x^2$$

Step 3
The next step is to determine the critical points for this equation.

$$A'(x) = 300 - 2x \qquad \to \qquad 300 - 2x = 0 \qquad \to \qquad x = 150$$

Step 4
Now for the step many neglect as unnecessary. Just because we got a single value we can't just assume that this will give a maximum product. We need to do a quick check to see if it does give a maximum.

As discussed in notes there are several methods for doing this, but in this case we can quickly see that,

$$A''(x) = -2$$

From this we can see that the second derivative is always negative and so $A(x)$ will always be concave down and so the single critical point we got in Step 3 must be a relative maximum and hence must be the value that gives a maximum product.

Step 5
Finally, let's actually answer the question. We need to give both values. We already have *x* so we need to determine *y* and that is easy to do from the constraint.

$$y = 300 - 150 = 150$$

The final answer is then,

$$\boxed{x = 150 \qquad y = 150}$$

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2. Find two positive numbers whose product is 750 and for which the sum of one and 10 times the other is a minimum.

Step 1
The first step is to write down equations describing this situation.

Let's call the two numbers *x* and *y* and we are told that the product is 750 (this is the constraint for the problem) or,

$$xy = 750$$

We are then being asked to minimize the sum of one and 10 times the other,

$$S = x + 10y$$

Note that it really doesn't worry which is *x* and which is *y* in the sum so we simply chose the *y* to be multiplied by 10.

Step 2
We now need to solve the constraint for *x* or *y* (and it really doesn't matter which variable we solve for in this case) and plug this into the product equation.

$$x = \frac{750}{y} \qquad \Rightarrow \qquad S(y) = \frac{750}{y} + 10y$$

Step 3
The next step is to determine the critical points for this equation.

$$S'(y) = -\frac{750}{y^2} + 10 \qquad \to \qquad -\frac{750}{y^2} + 10 = 0 \qquad \to \qquad y = \pm\sqrt{75} = 5\sqrt{3}$$

Because we are told that *y* must be positive we can eliminate the negative value and so the only value we really get out of this step is : $y = \sqrt{75} = 5\sqrt{3}$.

Step 4
Now for the step many neglect as unnecessary. Just because we got a single value we can't just assume that this will give a minimum sum. We need to do a quick check to see if it does give a minimum.

As discussed in notes there are several methods for doing this, but in this case we can quickly see that,

$$S''(y) = \frac{1500}{y^3}$$

From this we can see that, provided we recall that *y* is positive, then the second derivative will always be positive. Therefore, $S(y)$ will always be concave up and so the single critical point from Step 3 that we can use must be a relative minimum and hence must be the value that gives a minimum sum.

Step 5

Finally, let's actually answer the question. We need to give both values. We already have y so we need to determine x and that is easy to do from the constraint.

$$x = \frac{750}{5\sqrt{3}} = 50\sqrt{3}$$

The final answer is then,

$$\boxed{x = 50\sqrt{3} \qquad y = 5\sqrt{3}}$$

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3. Let x and y be two positive numbers such that $x + 2y = 50$ and $(x+1)(y+2)$ is a maximum.

Step 1
In this case we were given the constraint in the problem,

$$x + 2y = 50$$

We are also told the equation to maximize,

$$f = (x+1)(y+2)$$

So, let's just solve the constraint for x or y (we'll solve for x to avoid fractions…) and plug this into the product equation.

$$x = 50 - 2y \qquad \Rightarrow \qquad f(y) = (50 - 2y + 1)(y+2) = (51 - 2y)(y+2) = 102 + 47y - 2y^2$$

Step 2
The next step is to determine the critical points for this equation.

$$f'(y) = 47 - 4y \qquad \rightarrow \qquad 47 - 4y = 0 \qquad \rightarrow \qquad y = \frac{47}{4}$$

Step 3
Now for the step many neglect as unnecessary. Just because we got a single value we can't just assume that this will give a maximum product. We need to do a quick check to see if it does give a maximum.

As discussed in notes there are several methods for doing this, but in this case we can quickly see that,

$$f''(y) = -4$$

From this we can see that the second derivative is always negative and so $f(y)$ will always be concave down and so the single critical point we got in Step 2 must be a relative maximum and hence must be the value that gives a maximum.

Step 4
Finally, let's actually answer the question. We need to give both values. We already have *y* so we need to determine *x* and that is easy to do from the constraint.

$$x = 50 - 2\left(\frac{47}{4}\right) = \frac{53}{2}$$

The final answer is then,

$$\boxed{x = \frac{53}{2} \qquad y = \frac{47}{4}}$$

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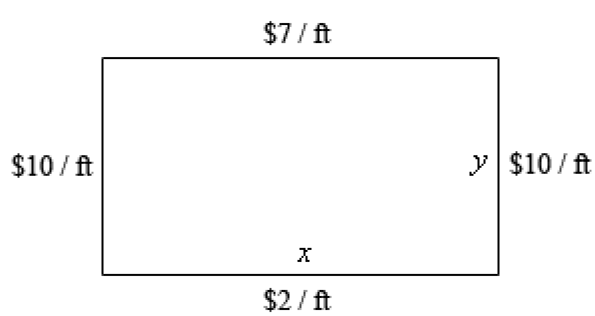
4. We are going to fence in a rectangular field. If we look at the field from above the cost of the vertical sides are \$10/ft, the cost of the bottom is \$2/ft and the cost of the top is \$7/ft. If we have \$700 determine the dimensions of the field that will maximize the enclosed area.

Step 1
The first step is to do a quick sketch of the problem. We could probably skip the sketch in this case, but that is a really bad habit to get into. For many of these problems a sketch is really convenient and it can be used to help us keep track of some of the important information in the problem and to "define" variables for the problem.

Here is the sketch for this problem.



Step 2
Next we need to set up the constraint and equation that we are being asked to optimize.

We are told that we have \$700 to spend and so the cost of the material will be the constraint for this problem. The cost for the material is then,

$$700 = 10y + 2x + 10y + 7x = 20y + 9x$$

We are being asked to maximize the area so that equation is,

$$A = xy$$

Step 3
Now, let's solve the constraint for y (that looks like it will only have one fraction in it and so may be "easier"…).

$$y = 35 - \tfrac{9}{20}x$$

Plugging this into the area formula gives,

$$A(x) = x\left(35 - \tfrac{9}{20}x\right) = 35x - \tfrac{9}{20}x^2$$

Step 4
Finding the critical point(s) for this shouldn't be too difficult at this point so here is that work.

$$A'(x) = 35 - \tfrac{9}{10}x \qquad \rightarrow \qquad 35 - \tfrac{9}{10}x = 0 \qquad \rightarrow \qquad x = \tfrac{350}{9}$$

Step 5
The second derivative of the area function is,

$$A''(x) = -\tfrac{9}{10}$$

From this we can see that the second derivative is always negative and so $A(x)$ will always be concave down and so the single critical point we got in Step 4 must be a relative maximum and hence must be the value that gives a maximum area.

Step 6
Now, let's finish the problem by getting the second dimension.

$$y = 35 - \tfrac{9}{20}\left(\tfrac{350}{9}\right) = \tfrac{35}{2}$$

The final dimensions are then,

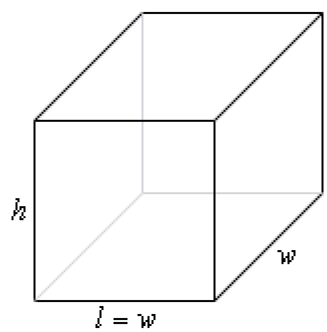
$$\boxed{x = \tfrac{350}{9} \qquad y = \tfrac{35}{2}}$$

5. We have 45 m^2 of material to build a box with a square base and no top. Determine the dimensions of the box that will maximize the enclosed volume.

Step 1

The first step is to do a quick sketch of the problem. We could probably skip the sketch in this case, but that is a really bad habit to get into. For many of these problems a sketch is really convenient and it can be used to help us keep track of some of the important information in the problem and to “define” variables for the problem.

Here is the sketch for this problem.



Step 2

Next we need to set up the constraint and equation that we are being asked to optimize.

We are told that we have 45 m^2 of material to build the box and so that is the constraint. The amount of material that we need to build the box is then,

$$45 = lw + 2\left(lh\right) + 2\left(wh\right) = w^2 + 2wh + 2wh = w^2 + 4wh$$

Note that because there is no top the first term won’t have the 2 that the second and third term have. Be careful with this kind of thing it is easy to miss if you aren’t paying attention.

We are being asked to maximize the volume so that equation is,

$$V = lwh = w^2 h$$

Note as well that we went ahead and used fact that $l = w$ in both of these equations to reduce the three variables in the equation down to two variables.

Step 3

Now, let's solve the constraint for *h* (that will allow us to avoid dealing with roots, plus there is only one *h* in the constraint so it will simply be easier to deal with).

$$h = \frac{45 - w^2}{4w}$$

Plugging this into the volume formula gives,

$$V(w) = w^2\left(\frac{45 - w^2}{4w}\right) = \frac{1}{4}w\left(45 - w^2\right) = \frac{1}{4}\left(45w - w^3\right)$$

Step 4

Finding the critical point(s) for this shouldn't be too difficult at this point so here is that work.

$$V'(w) = \tfrac{1}{4}\left(45 - 3w^2\right) \qquad \rightarrow \qquad \tfrac{1}{4}\left(45 - 3w^2\right) = 0 \qquad \rightarrow \qquad w = \pm\sqrt{\tfrac{45}{3}} = \pm\sqrt{15}$$

Because we are dealing with the dimensions of a box the negative width doesn't make any sense and so the only critical point that we can use here is : $w = \sqrt{15}$.

Be careful here and do not get into the habit of just eliminating the negative values. The only reason for eliminating it in this case is for physical reasons. If we had just given the equations without any physical reasoning it would have to be included in the rest of the work!

Step 5

The second derivative of the volume function is,

$$V''(w) = -6w$$

From this we can see that the second derivative is always negative for positive *w* (which we will always have for this case since *w* is the width of a box). Therefore, provided *w* is positive, $V(w)$ will always be concave down and so the single critical point we got in Step 4 must be a relative maximum and hence must be the value that gives a maximum volume.

Step 6

Now, let's finish the problem by getting the remaining dimensions.

$$l = w = \sqrt{15} = 3.8730 \qquad\qquad h = \frac{45 - 15}{4\sqrt{15}} = 1.9365$$

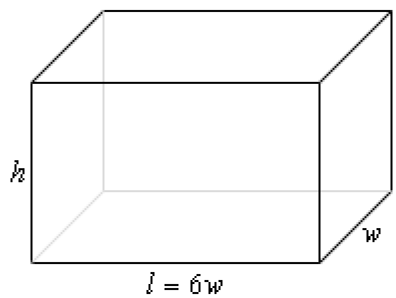
The final dimensions are then,

$$\boxed{l = w = 3.8730 \qquad h = 1.9365}$$

6. We want to build a box whose base length is 6 times the base width and the box will enclose 20 in^3. The cost of the material of the sides is \$3/in^2 and the cost of the top and bottom is \$15/in^2. Determine the dimensions of the box that will minimize the cost.

Step 1
The first step is to do a quick sketch of the problem. We could probably skip the sketch in this case, but that is a really bad habit to get into. For many of these problems a sketch is really convenient and it can be used to help us keep track of some of the important information in the problem and to "define" variables for the problem.

Here is the sketch for this problem.



Step 2
Next we need to set up the constraint and equation that we are being asked to optimize.

We are told that the volume of the box must be 20 in^3 and so this is the constraint.

$$20 = lwh = 6w^2 h$$

We are being asked to minimize the cost and the cost function is,

$$C = 3\left[2(lh) + 2(wh)\right] + 15\left[2(lw)\right] = 3\left[12wh + 2wh\right] + 15\left[12w^2\right] = 42wh + 180w^2$$

Note as well that we went ahead and used fact that $l = 6w$ in both of these equations to reduce the three variables in the equation down to two variables.

Step 3
Now, let's solve the constraint for h (that will allow us to avoid dealing with roots).

$$h = \frac{10}{3w^2}$$

Plugging this into the cost function gives,

$$C(w) = 42w\left(\frac{10}{3w^2}\right) + 180w^2 = \frac{140}{w} + 180w^2$$

Step 4
Finding the critical point(s) for this shouldn't be too difficult at this point. Here is the derivative.

$$C'(w) = -\frac{140}{w^2} + 360w = \frac{360w^3 - 140}{w^2}$$

From this it looks like the critical points are : $w = 0$ and $w = \sqrt[3]{\frac{7}{18}} = 0.7299$.
Because we are dealing with the dimensions of a box the zero width doesn't make any sense and so the only critical point that we can use here is : $w = \sqrt[3]{\frac{7}{18}} = 0.7299$.

Be careful here and do not get into the habit of just eliminating the zero as a critical point. The only reason for eliminating it in this case is for physical reasons. If we had just given the equations without any physical reasoning it would have to be included in the rest of the work!

Step 5
The second derivative of the volume function is,

$$C''(w) = \frac{280}{w^3} + 360$$

From this we can see that the second derivative is always positive for positive *w* (which we will always have for this case since *w* is the width of a box). Therefore, provided *w* is positive, $C(w)$ will always be concave up and so the single critical point we got in Step 4 must be a relative minimum and hence must be the value that gives a minimum cost.

Step 6
Now, let's finish the problem by getting the remaining dimensions.

$$l = 6w = 4.3794 \qquad\qquad h = \frac{10}{3(0.7299)^2} = 6.2568$$

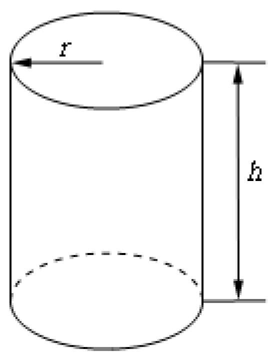
The final dimensions are then,

$$\boxed{w = 0.7299 \qquad l = 4.3794 \qquad h = 6.2568}$$

7. We want to construct a cylindrical can with a bottom but no top that will have a volume of 30 cm^3. Determine the dimensions of the can that will minimize the amount of material needed to construct the can.

Step 1
The first step is to do a quick sketch of the problem. We could probably skip the sketch in this case, but that is a really bad habit to get into. For many of these problems a sketch is really convenient and it can be used to help us keep track of some of the important information in the problem and to "define" variables for the problem.

Here is the sketch for this problem.



Step 2
Next we need to set up the constraint and equation that we are being asked to optimize.

We are told that the volume of the can must be 30 cm^3 and so this is the constraint.

$$30 = \pi r^2 h$$

We are being asked to minimize the amount of material needed to construct the can,

$$A = 2\pi r h + \pi r^2$$

Recall that the can will have no top and so the second term will only be for the area of the bottom of the can.

Step 3
Now, let's solve the constraint for h (that will allow us to avoid dealing with roots).

$$h = \frac{30}{\pi r^2}$$

Plugging this into the amount of material function gives,

$$A(r) = 2\pi r\left(\frac{30}{\pi r^2}\right) + \pi r^2 = \frac{60}{r} + \pi r^2$$

Step 4
Finding the critical point(s) for this shouldn't be too difficult at this point. Here is the derivative.

$$A'(r) = -\frac{60}{r^2} + 2\pi r = \frac{2\pi r^3 - 60}{r^2}$$

From this it looks like the critical points are : $r = 0$ and $r = \sqrt[3]{\frac{60}{2\pi}} = 2.1216$.
Because we are dealing with the dimensions of a can the zero radius doesn't make any sense and so the only critical point that we can use here is : $r = \sqrt[3]{\frac{60}{2\pi}} = 2.1216$.

Be careful here and do not get into the habit of just eliminating the zero as a critical point. The only reason for eliminating it in this case is for physical reasons. If we had just given the equations without any physical reasoning it would have to be included in the rest of the work!

Step 5
The second derivative of the volume function is,

$$A''(r) = \frac{120}{r^3} + 2\pi$$

From this we can see that the second derivative is always positive for positive *r* (which we will always have for this case since *r* is the radius of a can). Therefore, provided *r* is positive, $A(r)$ will always be concave up and so the single critical point we got in Step 4 must be a relative minimum and hence must be the value that gives a minimum amount of material.

Step 6
Now, let's finish the problem by getting the height of the can.

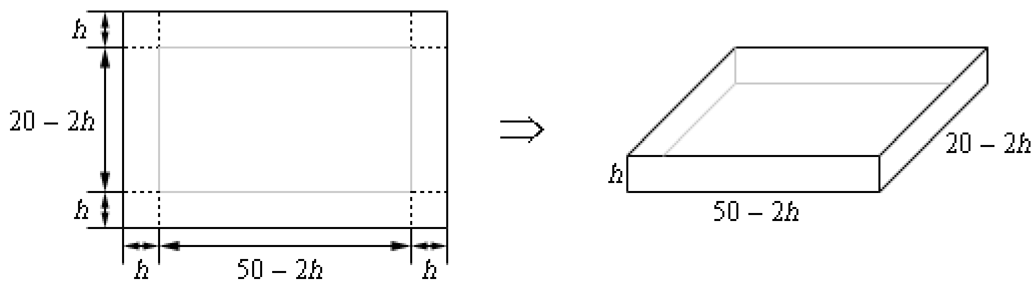
$$h = \frac{30}{\pi(2.1216)^2} = 2.1215$$

The final dimensions are then,

$$\boxed{r = 2.1216 \qquad\qquad h = 2.1215}$$

8. We have a piece of cardboard that is 50 cm by 20 cm and we are going to cut out the corners and fold up the sides to form a box. Determine the height of the box that will give a maximum volume.

Step 1
The first step is to do a quick sketch of the problem.



Step 2
As with the problem like this in the notes the constraint is really the size of the box and that has been taken into account in the figure so all we need to do is set up the volume equation that we want to maximize.

$$V(h) = h(50-2h)(20-2h) = 4h^3 - 140h^2 + 1000h$$

Step 3
Finding the critical point(s) for this shouldn't be too difficult at this point so here is that work,

$$V'(h) = 12h^2 - 280h + 1000 \qquad h = \tfrac{35 \pm 5\sqrt{19}}{3} = 4.4018, \quad 18.9315$$

From the figure above we can see that the limits on *h* must be $h = 0$ and $h = 10$ (the largest *h* could be is ½ the smaller side). Note that neither of these really make physical sense but they do provide limits on *h*.

So, we must have $0 \le h \le 10$ and this eliminates the second critical point and so the only critical point we need to worry about is $h = 4.4018$

Step 4
Because we have limits on *h* we can quickly check to see if we have maximum by plugging in the volume function.

$$V(0) = 0 \qquad V(4.4018) = 2030.34 \qquad V(10) = 0$$

So, we can see then that the height of the box will have to be $\boxed{h = 4.4018}$ in order to get a maximum volume.