Preface

Here are the solutions to the practice problems for my Calculus I notes. Some solutions will have more or less detail than other solutions. The level of detail in each solution will depend up on several issues. If the section is a review section, this mostly applies to problems in the first chapter, there will probably not be as much detail to the solutions given that the problems really should be review. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you've reached the level of working the harder problems then you will probably already understand the basics fairly well and won't need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven't been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

The Mean Value Theorem

1. Determine all the number(s) c which satisfy the conclusion of Rolle's Theorem for $f\left(x\right)=x^2-2x-8$ on $\left[-1,3\right]$.

Solution
The first thing we should do is actually verify that Rolle's Theorem can be used here.

The function is a polynomial which is continuous and differentiable everywhere and so will be continuous on $\left[-1,3\right]$ and differentiable on $\left(-1,3\right)$.

Next, a couple of quick function evaluations shows that $f\left(-1\right)=f\left(3\right)=-5$.

Therefore, the conditions for Rolle's Theorem are met and so we can actually do the problem.

Note that this may seem to be a little silly to check the conditions but it is a really good idea to get into the habit of doing this stuff. Since we are in this section it is pretty clear that the conditions will be met or we wouldn't be asking the problem. However, once we get out of this section and you want to use the Theorem the conditions may not be met. If you are in the habit of not checking you could inadvertently use the Theorem on a problem that can't be used and then get an incorrect answer.

Now that we know that Rolle's Theorem can be used there really isn't much to do. All we need to do is take the derivative,

$$f'(x) = 2x - 2$$

and then solve $f'(c) = 0$.

$$2c - 2 = 0 \qquad \Rightarrow \qquad c = 1$$

So, we found a single value and it is in the interval so the value we want is,

$$\boxed{c = 1}$$

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2. Determine all the number(s) c which satisfy the conclusion of Rolle's Theorem for $g(t) = 2t - t^2 - t^3$ on $[-2,1]$.

Solution
The first thing we should do is actually verify that Rolle's Theorem can be used here.

The function is a polynomial which is continuous and differentiable everywhere and so will be continuous on $[-2,1]$ and differentiable on $(-2,1)$.

Next, a couple of quick function evaluations shows that $g(-2) = g(1) = 0$.

Therefore, the conditions for Rolle's Theorem are met and so we can actually do the problem.

Note that this may seem to be a little silly to check the conditions but it is a really good idea to get into the habit of doing this stuff. Since we are in this section it is pretty clear that the conditions will be met or we wouldn't be asking the problem. However, once we get out of this section and you want to use the Theorem the conditions may not be met. If you are in the habit of not checking you could inadvertently use the Theorem on a problem that can't be used and then get an incorrect answer.

Now that we know that Rolle's Theorem can be used there really isn't much to do. All we need to do is take the derivative,

$$g'(t) = 2 - 2t - 3t^2$$

and then solve $g'(c) = 0$.

$$-3c^2 - 2c + 2 = 0 \qquad \Rightarrow \qquad c = \tfrac{1 \pm \sqrt{7}}{-3} = -1.2153,\ 0.5486$$

So, we found two values and, in this case, they are both in the interval so the values we want are,

$$\boxed{c = \tfrac{1 \pm \sqrt{7}}{-3} = -1.2153,\ 0.5486}$$

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3. Determine all the number(s) c which satisfy the conclusion of Mean Value Theorem for $h(z) = 4z^3 - 8z^2 + 7z - 2$ on $[2,5]$.

Solution
The first thing we should do is actually verify that the Mean Value Theorem can be used here.

The function is a polynomial which is continuous and differentiable everywhere and so will be continuous on $[2,5]$ and differentiable on $(2,5)$.

Therefore, the conditions for the Mean Value Theorem are met and so we can actually do the problem.

Note that this may seem to be a little silly to check the conditions but it is a really good idea to get into the habit of doing this stuff. Since we are in this section it is pretty clear that the conditions will be met or we wouldn't be asking the problem. However, once we get out of this section and you want to use the Theorem the conditions may not be met. If you are in the habit of not checking you could inadvertently use the Theorem on a problem that can't be used and then get an incorrect answer.

Now that we know that the Mean Value Theorem can be used there really isn't much to do. All we need to do is do some function evaluations and take the derivative.

$$h(2) = 12 \qquad h(5) = 333 \qquad\qquad h'(z) = 12z^2 - 16z + 7$$

The final step is to then plug into the formula from the Mean Value Theorem and solve for c.

$$12c^2 - 16c + 7 = \tfrac{333-12}{5-2} = 107 \qquad \to \qquad 12c^2 - 16c - 100 = 0$$
$$c = \tfrac{2 \pm \sqrt{79}}{3} = -2.2961, \quad 3.6294$$

So, we found two values and, in this case, only the second is in the interval and so the value we want is,

$$\boxed{c = \tfrac{2+\sqrt{79}}{3} = 3.6294}$$

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4. Determine all the number(s) c which satisfy the conclusion of Mean Value Theorem for $A(t) = 8t + \mathbf{e}^{-3t}$ on $[-2,3]$.

Solution
The first thing we should do is actually verify that the Mean Value Theorem can be used here.

The function is a sum of a polynomial and an exponential function both of which are continuous and differentiable everywhere. This in turn means that the sum is also continuous and differentiable everywhere and so the function will be continuous on $[-2,3]$ and differentiable on $(-2,3)$.

Therefore, the conditions for the Mean Value Theorem are met and so we can actually do the problem.

Note that this may seem to be a little silly to check the conditions but it is a really good idea to get into the habit of doing this stuff. Since we are in this section it is pretty clear that the conditions will be met or we wouldn't be asking the problem. However, once we get out of this section and you want to use the Theorem the conditions may not be met. If you are in the habit of not checking you could inadvertently use the Theorem on a problem that can't be used and then get an incorrect answer.

Now that we know that the Mean Value Theorem can be used there really isn't much to do. All we need to do is do some function evaluations and take the derivative.

$$A(-2) = -16 + \mathbf{e}^{6} \qquad A(3) = 24 + \mathbf{e}^{-9} \qquad\qquad A'(t) = 8 - 3\mathbf{e}^{-3t}$$

The final step is to then plug into the formula from the Mean Value Theorem and solve for c.

$$\begin{aligned} 8 - 3\mathbf{e}^{-3c} &= \tfrac{24+\mathbf{e}^{-9}-\left(-16+\mathbf{e}^{6}\right)}{3-(-2)} = -72.6857 \\ 3\mathbf{e}^{-3c} &= 80.6857 \\ \mathbf{e}^{-3c} &= 26.8952 \\ -3c &= \ln\left(26.8952\right) = 3.29195 \qquad \Rightarrow \qquad c = -1.0973 \end{aligned}$$

So, we found a single value and it is in the interval and so the value we want is,

$$\boxed{c = -1.0973}$$

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5. Suppose we know that $f(x)$ is continuous and differentiable on the interval $[-7,0]$, that $f(-7) = -3$ and that $f'(x) \le 2$. What is the largest possible value for $f(0)$?

Step 1
We were told in the problem statement that the function (whatever it is) satisfies the conditions of the Mean Value Theorem so let's start out this that and plug in the known values.

$$f(0) - f(-7) = f'(c)(7-0) \qquad \rightarrow \qquad f(0) + 3 = 7f'(c)$$

Step 2
Next, let's solve for $f(0)$.

$$f(0) = 7f'(c) - 3$$

Step 3
Finally, let's take care of what we know about the derivative. We are told that the maximum value of the derivative is 2. So, plugging the maximum possible value of the derivative into $f'(c)$ above will, in this case, give us the maximum value of $f(0)$. Doing this gives,

$$f(0) = 7f'(c) - 3 \le 7(2) - 3 = 11$$

So, the largest possible value for $f(0)$ is 11. Or, written as an inequality this would be written as,

$$\boxed{f(0) \le 11}$$

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6. Show that $f(x) = x^3 - 7x^2 + 25x + 8$ has exactly one real root.

Hint : Can you use the Intermediate Value Theorem to prove that it has at least one real root?

Step 1

First let's note that $f(0) = 8$. If we could find a function value that was negative the Intermediate Value Theorem (which can be used here because the function is continuous everywhere) would tell us that the function would have to be zero somewhere. In other words, there would have to be at least one real root.

Because the largest power of x is 3 it looks like if we let x be large enough and negative the function should also be negative. All we need to do is start plugging in negative x's until we find one that works. In fact, we don't even need to do much : $f(-1) = -25$.

So, we can see that $-25 = f(-1) < 0 < f(0) = 8$ and so by the Intermediate Value Theorem the function must be zero somewhere in the interval $(-1, 0)$. The interval itself is not important. What is important is that we have at least one real root.

Hint : What would happen if there were more than one real root?

Step 2
Next, let's assume that there is more than one real root. Assuming this means that there must be two numbers, say a and b, so that,

$$f(a) = f(b) = 0$$

Next, because $f(x)$ is a polynomial it is continuous and differentiable everywhere and so we could use Rolle's Theorem to see that there must be a real value, c, so that,

$$f'(c) = 0$$

Note that Rolle's Theorem tells us that c must be between a and b. Since both of these are real values then c must also be real.

Hint : Is that possible?

Step 3
Because $f(x)$ is a polynomial it is easy enough to see if such a c exists.

$$f'(x) = 3x^2 - 14x + 25 \qquad \to \qquad 3c^2 - 14c + 25 = 0 \qquad \to \qquad c = \frac{7 \pm \sqrt{26}\, i}{3}$$

So, we can see that in fact the only two places where the derivative is zero are complex numbers are not real numbers. Therefore, it is not possible for there to be more than one real root.

From Step 1 we know that there is at least one real root and we've just proven that we can't have more than one real root. Therefore, there must be **exactly one real root**.
