CALCULUS I

Solutions to Practice Problems
Applications of Integrals

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Preface

Here are the solutions to the practice problems for my Calculus I notes. Some solutions will have more or less detail than other solutions. The level of detail in each solution will depend up on several issues. If the section is a review section, this mostly applies to problems in the first chapter, there will probably not be as much detail to the solutions given that the problems really should be review. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you've reached the level of working the harder problems then you will probably already understand the basics fairly well and won't need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven't been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

Applications of Integrals

Average Function Value

1. Determine f_{avg} for $f(x) = 8x - 3 + 5\mathbf{e}^{2-x}$ on $[0,2]$.

Solution

There really isn't all that much to this problem other than use the formula given in the notes for this section.

$$f_{\text{avg}} = \tfrac{1}{2-0}\int_0^2 8x - 3 + 5\mathbf{e}^{2-x}\,dx = \tfrac{1}{2}\left(4x^2 - 3x - 5\mathbf{e}^{2-x}\right)\Big|_0^2 = \boxed{\tfrac{1}{2}\left(5 + 5\mathbf{e}^2\right)}$$

Note that we are assuming your integration skills are pretty good at this point and won't be showing many details of the actual integration process. This includes not showing substitutions such as the substitution needed for the third term (you did catch that correct?).

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2. Determine f_{avg} for $f(x) = \cos(2x) - \sin\left(\tfrac{x}{2}\right)$ on $\left[-\tfrac{\pi}{2}, \pi\right]$.

Solution

There really isn't all that much to this problem other than use the formula given in the notes for this section.

$$f_{\text{avg}} = \tfrac{1}{\pi - \left(-\tfrac{\pi}{2}\right)}\int_{-\tfrac{\pi}{2}}^{\pi} \cos(2x) - \sin\left(\tfrac{x}{2}\right)dx = \tfrac{2}{3\pi}\left(\tfrac{1}{2}\sin(2x) + 2\cos\left(\tfrac{x}{2}\right)\right)\Big|_{-\tfrac{\pi}{2}}^{\pi} = \boxed{-\tfrac{2\sqrt{2}}{3\pi}}$$

Note that we are assuming your integration skills are pretty good at this point and won't be showing many details of the actual integration process. This includes not showing either of the substitutions needed for the integral (you did catch both of them correct?).

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3. Find f_{avg} for $f(x) = 4x^2 - x + 5$ on $[-2,3]$ and determine the value(s) of c in $[-2,3]$ for which $f(c) = f_{\text{avg}}$.

Step 1
First we need to use the formula for the notes in this section to find f_{avg}.

$$f_{\text{avg}} = \tfrac{1}{3-(-2)}\int_{-2}^3 4x^2 - x + 5\,dx = \tfrac{1}{5}\left(\tfrac{4}{3}x^3 - \tfrac{1}{2}x^2 + 5x\right)\Big|_{-2}^3 = \boxed{\tfrac{83}{6}}$$

Step 2
Note that for the second part of this problem we are really just asking to find the value of c that satisfies the Mean Value Theorem for Integrals.

There really isn't much to do here other than solve $f(c) = f_{\text{avg}}$.

$$4c^2 - c + 5 = \frac{83}{6}$$

$$4c^2 - c - \frac{53}{6} = 0 \qquad \Rightarrow \qquad c = \frac{1 \pm \sqrt{1 - 4(4)\left(-\frac{53}{6}\right)}}{2(4)} = \frac{1 \pm \sqrt{\frac{427}{3}}}{2(4)} = \boxed{-1.3663, \ 1.6163}$$

So, unlike the example from the notes both of the numbers that we found here are in the interval and so are both included in the answer.

Area Between Curves

1. Determine the area below $f(x) = 3 + 2x - x^2$ and above the *x*-axis.

Hint : It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the upper/lower functions and the limits for the integral.

Step 1
Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.
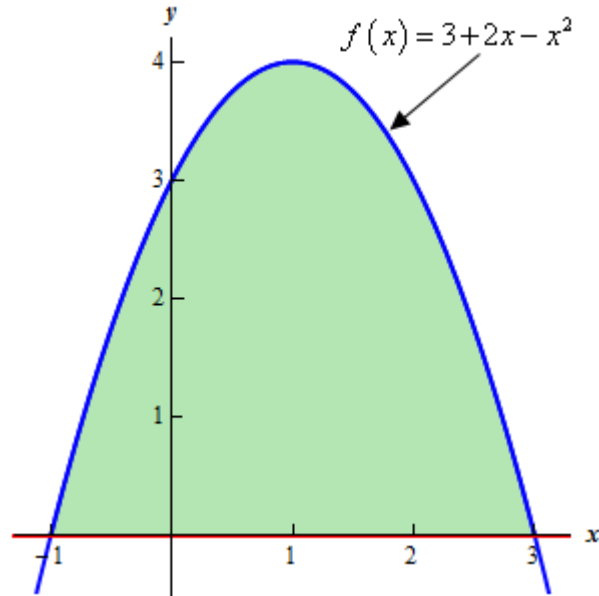
Step 2

It should be clear from the graph that the upper function is the parabola (*i.e.* $y = 3 + 2x - x^2$) and the lower function is the *x*-axis (*i.e.* $y = 0$).

Since we weren't given any limits on *x* in the problem statement we'll need to get those. From the graph it looks like the limits are (probably) $-1 \le x \le 3$. However, we should never just assume that our graph is accurate or that we were able to read it accurately. For all we know the limits are close to those we guessed from the graph but are in fact slightly different.

So, to determine if we guessed the limits correctly from the graph let's find them directly. The limits are where the parabola crosses the *x*-axis and so all we need to do is set the parabola equal to zero (*i.e.* where it crosses the line $y = 0$) and solve. Doing this gives,

$$3 + 2x - x^2 = 0 \quad \rightarrow \quad -(x+1)(x-3) = 0 \quad \rightarrow \quad x = -1, \ x = 3$$

So, we did guess correctly, but it never hurts to be sure. That is especially true here where finding them directly takes almost no time.

Step 3

At this point there isn't much to do other than step up the integral and evaluate it.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$A = \int_{-1}^{3} 3 + 2x - x^2 \, dx = \left(3x + x^2 - \tfrac{1}{3}x^3\right)\Big|_{-1}^{3} = \boxed{\tfrac{32}{3}}$$

2. Determine the area to the left of $g(y) = 3 - y^2$ and to the right of $x = -1$.

Hint : It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the right/left functions and the limits for the integral.

Step 1
Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.
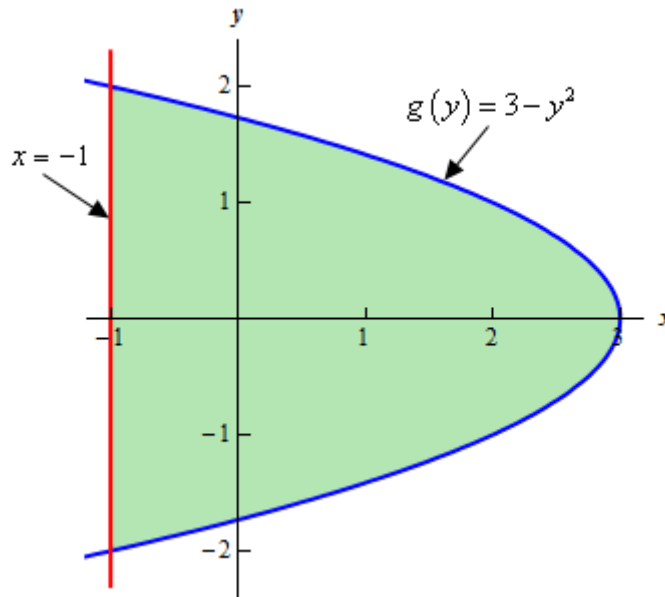
Step 2
It should be clear from the graph that the right function is the parabola (*i.e.* $x = 3 - y^2$) and the left function is the line $x = -1$.

Since we weren't given any limits on *y* in the problem statement we'll need to get those. However, we should never just assume that our graph is accurate or that we will be able to read it accurately enough to guess the limits from the graph. This is especially true when the

intersection points of the two curves (*i.e.* the limits on *y* that we need) do not occur on an axis (as they don't in this case).

So, to determine the intersection points correctly we'll need to find them directly. The intersection points are where the two curves intersect and so all we need to do is set the two equations equal and solve. Doing this gives,

$$3 - y^2 = -1 \quad \rightarrow \quad y^2 = 4 \quad \rightarrow \quad y = -2, \; y = 2$$

So, the limits on *y* are : $-2 \le y \le 2$.

Step 3
At this point there isn't much to do other than step up the integral and evaluate it.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$A = \int_{-2}^{2} 3 - y^2 - \left(-1\right) dy = \int_{-2}^{2} 4 - y^2 dy = \left(4y - \tfrac{1}{3} y^3\right)\Big|_{-2}^{2} = \boxed{\tfrac{32}{3}}$$

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3. Determine the area of the region bounded by $y = x^2 + 2$, $y = \sin(x)$, $x = -1$ and $x = 2$.

Hint : It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the upper/lower functions and the limits for the integral.

Step 1
Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.
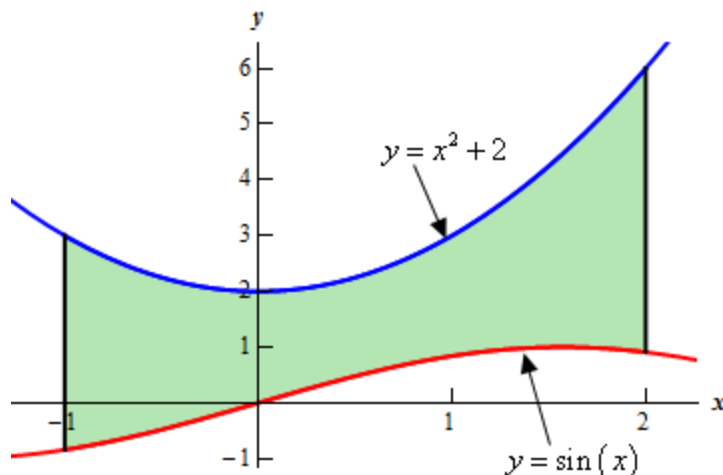
Step 2

It should be clear from the graph that the upper function is $y = x^2 + 2$ and the lower function is $y = \sin(x)$.

Next, we were given limits on x in the problem statement and we can see that the two curves do not intersect in that range. Note that this is something that we can't always guarantee and so we need the graph to verify if the curves intersect or not. We should never just assume that because limits on x were given in the problem statement that the curves will not intersect anywhere between the given limits.

So, because the curves do not intersect we will be able to find the area with a single integral using the limits : $-1 \le x \le 2$.

Step 3

At this point there isn't much to do other than step up the integral and evaluate it.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$A = \int_{-1}^{2} x^2 + 2 - \sin(x)\, dx = \left(\tfrac{1}{3}x^3 + 2x + \cos(x) \right)\Big|_{-1}^{2} = \boxed{9 + \cos(2) - \cos(1) = 8.04355}$$

Don't forget to set your calculator to radians if you take the answer to a decimal.

4. Determine the area of the region bounded by $y = \dfrac{8}{x}$, $y = 2x$ and $x = 4$.

Hint : It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the upper/lower functions and the limits for the integral.

Step 1
Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.
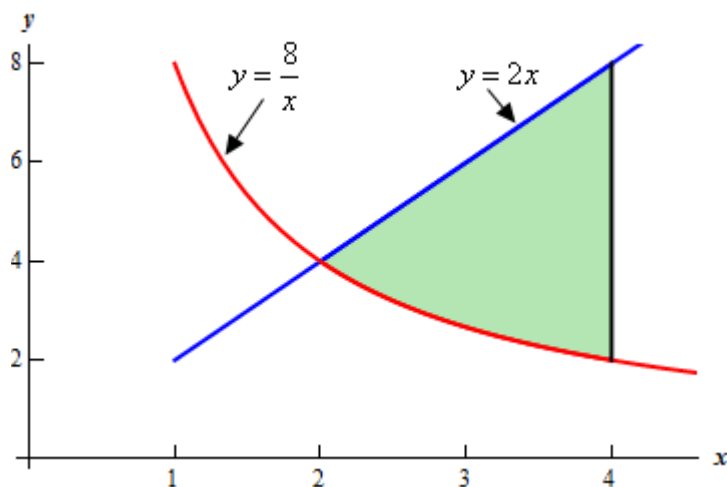
For this problem we were only given one limit on x (*i.e.* $x = 4$). To determine just what the region we are after recall that we are after a *bounded* region. This means that one of the given curves must be on each boundary of the region.

Therefore we can't use any portion of the region to the right of the line $x = 4$ because there will never be a boundary on the right of that region.

We also can't take any portion of the region to the left of the intersection point. Because the first function is not continuous at $x = 0$ we can't use any region that includes $x = 0$. Therefore any portion of the region to the left of the intersection point would have to stop prior to the y-axis and any region like that would not have any of the given curves on the left boundary.

The region is then the one shown in graph above. We will take the region to the left of the line $x = 4$ and to the right of the intersection point.

Step 2

We now need to determine the intersection point. However, we should never just assume that our graph is accurate or that we will be able to read it accurately enough to guess the coordinates from the graph. This is especially true when the intersection point of the two curves does not occur on an axis (as they don't in this case).

So, to determine the intersection point correctly we'll need to find it directly. The intersection point is where the two curves intersect and so all we need to do is set the two equations equal and solve. Doing this gives,

$$\frac{8}{x} = 2x \quad \rightarrow \quad x^2 = 4 \quad \rightarrow \quad x = -2, \ x = 2$$

Note that while we got two answers here the negative value does not make any sense because to get to that value we would have to go through $x = 0$ and as we discussed above the bounded region cannot contain $x = 0$.

Therefore the limits on x are : $2 \le x \le 4$.

It should also be clear from the graph and the limits above that the upper function is $y = 2x$ and the lower function is $y = \dfrac{8}{x}$.

Step 3

At this point there isn't much to do other than step up the integral and evaluate it.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$A = \int_{2}^{4} 2x - \frac{8}{x} \, dx = \left(x^2 - 8\ln|x| \right)\Big|_{2}^{4} = \boxed{12 - 8\ln(4) + 8\ln(2) = 6.4548}$$

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5. Determine the area of the region bounded by $x = 3 + y^2$, $x = 2 - y^2$, $y = 1$ and $y = -2$.

Hint : It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the right/left functions and the limits for the integral.

Step 1

Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.
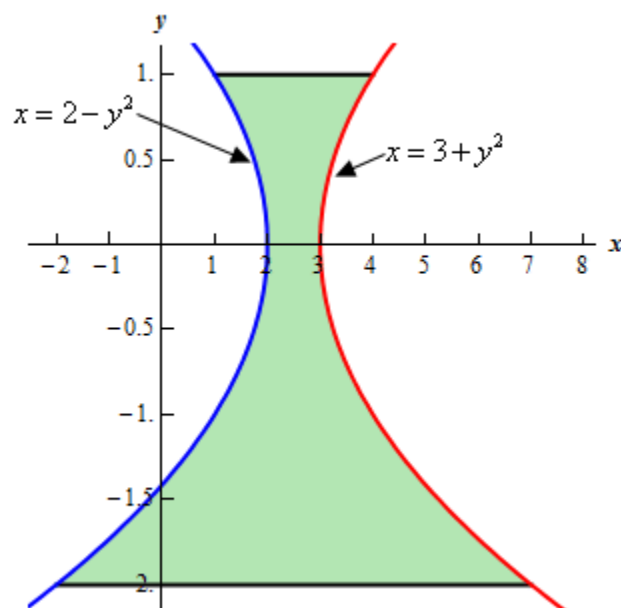
Step 2

It should be clear from the graph that the right function is $x = 3 + y^2$ and the left function is $x = 2 - y^2$.

Next, we were given limits on y in the problem statement and we can see that the two curves do not intersect in that range. Note that this is something that we can't always guarantee and so we need the graph to verify if the curves intersect or not. We should never just assume that because limits on y were given in the problem statement that the curves will not intersect anywhere between the given limits.

So, because the curves do not intersect we will be able to find the area with a single integral using the limits : $-2 \le y \le 1$.

Step 3

At this point there isn't much to do other than step up the integral and evaluate it.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$A = \int_{-2}^{1} 3 + y^2 - \left(2 - y^2\right) dy = \int_{-2}^{1} 1 + 2y^2 \, dy = \left(y + \tfrac{2}{3}y^3\right)\Big|_{-2}^{1} = \boxed{9}$$

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6. Determine the area of the region bounded by $x = y^2 - y - 6$ and $x = 2y + 4$.

Hint : It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the right/left functions and the limits for the integral.

Step 1
Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.
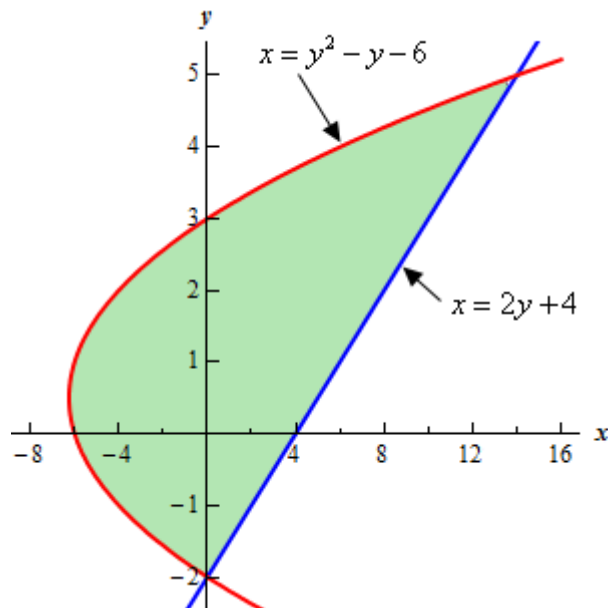
Note that we won't include any portion of the region above the top intersection point or below the bottom intersection point. The region needs to be bounded by one of the given curves on each

boundary. If we went past the top intersection point we would not have an upper bound on the region. Likewise, if we went past the bottom intersection point we would not have a lower bound on the region.

Step 2
It should be clear from the graph that the right function is $x = 2y + 4$ and the left function is $x = y^2 - y - 6$.

Since we weren't given any limits on y in the problem statement we'll need to get those. However, we should never just assume that our graph is accurate or that we will be able to read it accurately enough to guess the coordinates from the graph. This is especially true when the intersection points of the two curves (*i.e.* the limits on y that we need) do not occur on an axis (as they don't in this case).

So, to determine the intersection points correctly we'll need to find them directly. The intersection points are where the two curves intersect and so all we need to do is set the two equations equal and solve. Doing this gives,

$$y^2 - y - 6 = 2y + 4 \quad \rightarrow \quad y^2 - 3y - 10 = (y - 5)(y + 2) = 0 \quad \rightarrow \quad y = -2, \ y = 5$$

Therefore the limits on y are : $-2 \le y \le 5$.

Note that you may well have found the intersection points in the first step to help with the graph if you were graphing by hand which is not a bad idea with faced with graphing this kind of region.

Step 3
At this point there isn't much to do other than step up the integral and evaluate it.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$A = \int_{-2}^{5} 2y + 4 - \left(y^2 - y - 6\right) dy = \int_{-2}^{5} 10 + 3y - y^2 \, dy = \left(10y + \tfrac{3}{2} y^2 - \tfrac{1}{3} y^3\right)\Big|_{-2}^{5} = \boxed{\tfrac{343}{6}}$$

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7. Determine the area of the region bounded by $y = x\sqrt{x^2 + 1}$, $y = e^{-\frac{1}{2}x}$, $x = -3$ and the y-axis.
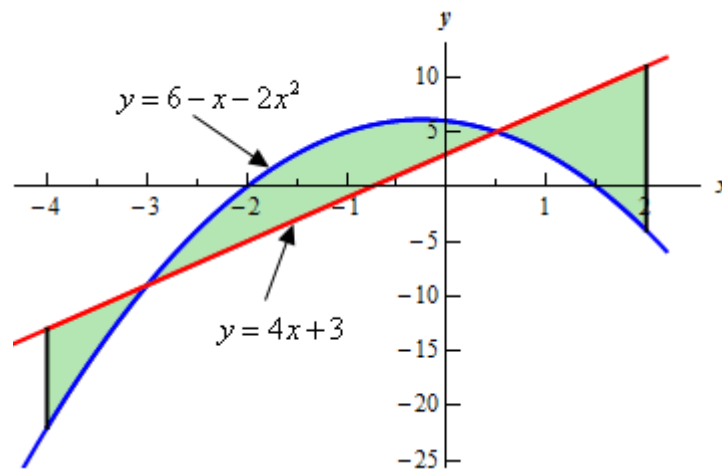
Hint : It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the upper/lower functions and the limits for the integral.

Step 1
Let's start off with getting a sketch of the region we want to find the area of.

Note that using a graphing calculator or computer may be needed to deal with the first equation, however you should be able to sketch the graph of the second equation by hand.

Here is a sketch of the bounded region we want to find the area of.
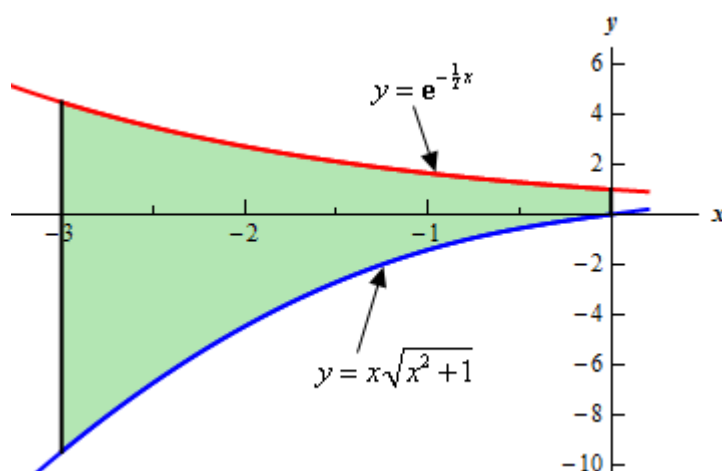
Step 2

It should be clear from the graph that the upper function is $y = e^{-\frac{1}{2}x}$ and the lower function is $y = x\sqrt{x^2 + 1}$.

Next, we were given limits on x in the problem statement (recall that the y-axis is just the line $x = 0$!) and we can see that the two curves do not intersect in that range. Note that this is something that we can't always guarantee and so we need the graph to verify if the curves intersect or not. We should never just assume that because limits on x were given in the problem statement that the curves will not intersect anywhere between the given limits.

So, because the curves do not intersect we will be able to find the area with a single integral using the limits : $-3 \le x \le 0$.

Step 3
At this point there isn't much to do other than step up the integral and evaluate it.

We are assuming that you are comfortable with basic integration techniques, including substitution since that will be needed here, so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$A = \int_{-3}^{0} e^{-\frac{1}{2}x} - x\sqrt{x^2+1}\, dx = \left(-2e^{-\frac{1}{2}x} - \frac{1}{3}(x^2+1)^{\frac{3}{2}}\right)\Big|_{-3}^{0} = \boxed{-\frac{7}{3} + 2e^{\frac{3}{2}} + \frac{1}{3}10^{\frac{3}{2}} = 17.17097}$$

8. Determine the area of the region bounded by $y = 4x + 3$, $y = 6 - x - 2x^2$, $x = -4$ and $x = 2$.

Hint : It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the upper/lower functions and the limits for the integral.

Step 1
Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.

Step 2
In the problem statement we were given two limits on x. However, as seen in the sketch of the graph above the curves intersect in this region and the upper/lower functions differ depending on what range of x's we are looking for.

Therefore we'll need to find the intersection points. However, we should never just assume that our graph is accurate or that we will be able to read it accurately enough to guess the coordinates from the graph. This is especially true when the intersection points of the two curves do not occur on an axis (as they don't in this case).

So, to determine the intersection points correctly we'll need to find them directly. The intersection points are where the two curves intersect and so all we need to do is set the two equations equal and solve. Doing this gives,

$$6 - x - 2x^2 = 4x + 3 \quad \rightarrow \quad 2x^2 + 5x - 3 = (2x - 1)(x + 3) = 0 \quad \rightarrow \quad x = -3, \ x = \tfrac{1}{2}$$

Note that you may well have found the intersection points in the first step to help with the graph if you were graphing by hand which is not a bad idea with faced with graphing this kind of region.

So, from the graph then it looks like we'll need three integrals since there are three ranges of x ($-4 \le x \le -3$, $-3 \le x \le \tfrac{1}{2}$ and $\tfrac{1}{2} \le x \le 2$) for which the upper/lower functions are different.

Step 3
At this point there isn't much to do other than step up the integrals and evaluate them.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$
\begin{aligned}
A &= \int_{-4}^{-3} 4x + 3 - \left(6 - x - 2x^2\right) dx + \int_{-3}^{\frac{1}{2}} 6 - x - 2x^2 - \left(4x + 3\right) dx + \int_{\frac{1}{2}}^{2} 4x + 3 - \left(6 - x - 2x^2\right) dx \\
&= \int_{-4}^{-3} 2x^2 + 5x - 3\, dx + \int_{-3}^{\frac{1}{2}} 3 - 5x - 2x^2\, dx + \int_{\frac{1}{2}}^{2} 2x^2 + 5x - 3\, dx \\
&= \left(\tfrac{2}{3} x^3 + \tfrac{5}{2} x^2 - 3x\right)\Big|_{-4}^{-3} + \left(3x - \tfrac{5}{2} x^2 - \tfrac{2}{3} x^3\right)\Big|_{-3}^{\frac{1}{2}} + \left(\tfrac{2}{3} x^3 + \tfrac{5}{2} x^2 - 3x\right)\Big|_{\frac{1}{2}}^{2} \\
&= \tfrac{25}{6} + \tfrac{343}{24} + \tfrac{81}{8} = \boxed{\tfrac{343}{12}}
\end{aligned}
$$

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9. Determine the area of the region bounded by $y = \dfrac{1}{x+2}$, $y = (x+2)^2$, $x = -\dfrac{3}{2}$, $x = 1$.
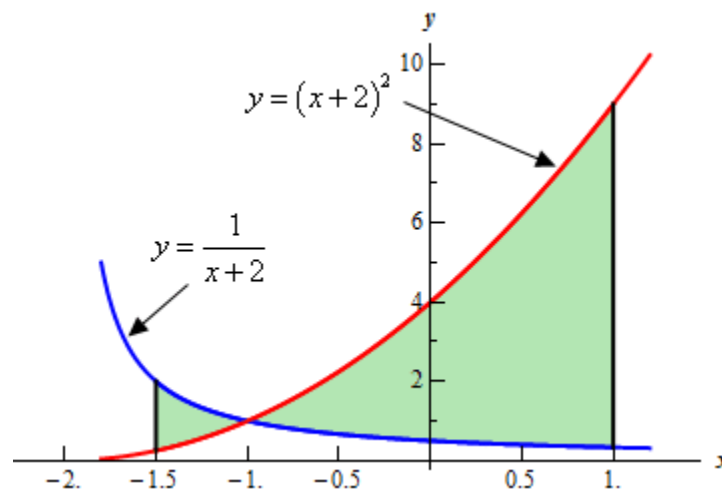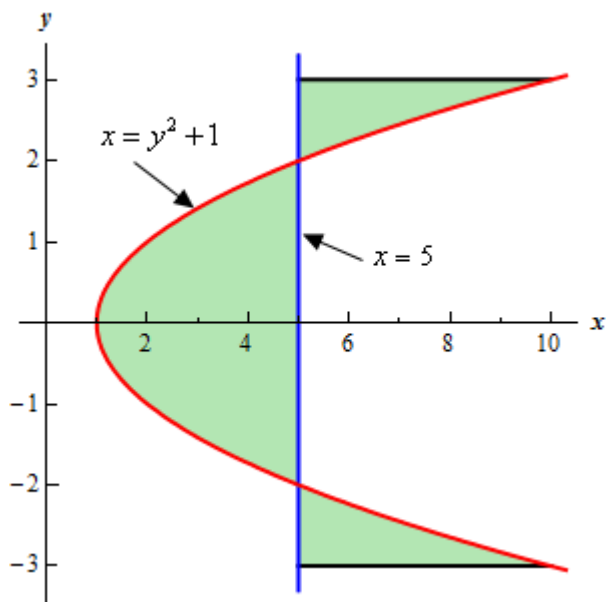
Hint : It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the upper/lower functions and the limits for the integral.

Step 1
Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.

Step 2
In the problem statement we were given two limits on x. However, as seen in the sketch of the graph above the curves intersect in this region and the upper/lower functions differ depending on what range of x's we are looking for.

Therefore we'll need to find the intersection point. However, we should never just assume that our graph is accurate or that we will be able to read it accurately enough to guess the coordinates from the graph. This is especially true when the intersection point of the two curves does not occur on an axis (as they don't in this case).

So, to determine the intersection points correctly we'll need to find it directly. The intersection point is where the two curves intersect and so all we need to do is set the two equations equal and solve. Doing this gives,

$$\frac{1}{x+2} = (x+2)^2 \quad \rightarrow \quad (x+2)^3 = 1 \quad \rightarrow \quad x+2 = \sqrt[3]{1} = 1 \quad \rightarrow \quad x = -1$$

So, from the graph then it looks like we'll need two integrals since there are two ranges of x ($-\frac{3}{2} \le x \le -1$ and $-1 \le x \le 1$) for which the upper/lower functions are different.

Step 3
At this point there isn't much to do other than step up the integrals and evaluate them.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$
\begin{aligned}
A &= \int_{-\frac{3}{2}}^{-1} \frac{1}{x+2} - (x+2)^2 \, dx + \int_{-1}^{1} (x+2)^2 - \frac{1}{x+2} \, dx \\
&= \left(\ln|x+2| - \tfrac{1}{3}(x+2)^3 \right)\Big|_{-\frac{3}{2}}^{-1} + \left(\tfrac{1}{3}(x+2)^3 - \ln|x+2| \right)\Big|_{-1}^{1} \\
&= \left[-\tfrac{7}{24} - \ln\left(\tfrac{1}{2}\right) \right] + \left[\tfrac{26}{3} - \ln(3) \right] = \boxed{\tfrac{67}{8} - \ln\left(\tfrac{1}{2}\right) - \ln(3) = 7.9695}
\end{aligned}
$$

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10. Determine the area of the region bounded by $x = y^2 + 1$, $x = 5$, $y = -3$ and $y = 3$.

Hint : It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the right/left functions and the limits for the integral.

Step 1
Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.

Step 2
In the problem statement we were given two limits on *y*. However, as seen in the sketch of the graph above the curves intersect in this region and the right/left functions differ depending on what range of *y*'s we are looking for.

Therefore we'll need to find the intersection points. However, we should never just assume that our graph is accurate or that we will be able to read it accurately enough to guess the coordinates from the graph. This is especially true when the intersection points of the two curves do not occur on an axis (as they don't in this case).

So, to determine the intersection points correctly we'll need to find them directly. The intersection points are where the two curves intersect and so all we need to do is set the two equations equal and solve. Doing this gives,

$$y^2 + 1 = 5 \quad \rightarrow \quad y^2 = 4 \quad \rightarrow \quad y = -2, \ y = 2$$

Note that you may well have found the intersection points in the first step to help with the graph if you were graphing by hand which is not a bad idea with faced with graphing this kind of region.

So, from the graph then it looks like we'll need three integrals since there are three ranges of *x* ($-3 \le x \le -2$, $-2 \le x \le 2$ and $2 \le x \le 3$) for which the right/left functions are different.

Step 3
At this point there isn't much to do other than step up the integrals and evaluate them.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$
\begin{aligned}
A &= \int_{-3}^{-2} y^2 + 1 - 5 \, dy + \int_{-2}^{2} 5 - \left(y^2 + 1 \right) dy + \int_{2}^{3} y^2 + 1 - 5 \, dy \\
&= \int_{-3}^{-2} y^2 - 4 \, dy + \int_{-2}^{2} 4 - y^2 \, dy + \int_{2}^{3} y^2 - 4 \, dy \\
&= \left(\tfrac{1}{3} y^3 - 4 y \right) \Big|_{-3}^{-2} + \left(4 y - \tfrac{1}{3} y^3 \right) \Big|_{-2}^{2} + \left(\tfrac{1}{3} y^3 - 4 y \right) \Big|_{2}^{3} \\
&= \tfrac{7}{3} + \tfrac{32}{3} + \tfrac{7}{3} = \boxed{\tfrac{46}{3}}
\end{aligned}
$$

---

11. Determine the area of the region bounded by $x = \mathbf{e}^{1+2y}$, $x = \mathbf{e}^{1-y}$, $y = -2$ and $y = 1$.

Hint : It's generally best to sketch the bounded region that we want to find the area of before starting the actual problem. Having the sketch of the graph will usually help with determining the right/left functions and the limits for the integral.

Step 1
Let's start off with getting a sketch of the region we want to find the area of.

We are assuming that, at this point, you are capable of graphing most of the basic functions that we're dealing with in these problems and so we won't be showing any of the graphing work here.

Here is a sketch of the bounded region we want to find the area of.

Step 2

In the problem statement we were given two limits on y. However, as seen in the sketch of the graph above the curves intersect in this region and the right/left functions differ depending on what range of y's we are looking for.

Therefore we'll need to find the intersection point. However, we should never just assume that our graph is accurate or that we will be able to read it accurately enough to guess the coordinates from the graph. In this case it seems pretty clear from the graph that the intersection point lies on the x-axis (and so we can guess the point we need is $y = 0$). However, for all we know the actual intersection point is slightly above or slightly below the x-axis and the scale of the graph just makes this hard to see.

So, to determine the intersection points correctly we'll need to find it directly. The intersection point is where the two curves intersect and so all we need to do is set the two equations equal and solve. Doing this gives,

$$ \mathbf{e}^{1+2y} = \mathbf{e}^{1-y} \quad \rightarrow \quad \frac{\mathbf{e}^{1+2y}}{\mathbf{e}^{1-y}} = 1 \quad \rightarrow \quad \mathbf{e}^{3y} = 1 \quad \rightarrow \quad y = 0 $$

So, from the graph then it looks like we'll need two integrals since there are two ranges of x ($-2 \le x \le 0$ and $0 \le x \le 1$) for which the right/left functions are different.

Step 3

At this point there isn't much to do other than step up the integrals and evaluate them.

We are assuming that you are comfortable with basic integration techniques so we'll not be including any discussion of the actual integration process here and we will be skipping some of the intermediate steps.

The area is,

$$A = \int_{-2}^{0} e^{1-y} - e^{1+2y} \, dy + \int_{0}^{1} e^{1+2y} - e^{1-y} \, dy$$

$$= \left(-e^{1-y} - \tfrac{1}{2} e^{1+2y} \right) \Big|_{-2}^{0} + \left(\tfrac{1}{2} e^{1+2y} + e^{1-y} \right) \Big|_{0}^{1}$$

$$= \left[e^3 + \tfrac{1}{2} e^{-3} - \tfrac{3}{2} e \right] + \left[1 + \tfrac{1}{2} e^3 - \tfrac{3}{2} e \right] = \boxed{22.9983}$$

Volumes of Solids of Revolution / Method of Rings

1. Use the method of disks/rings to determine the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $y = 3$ and the y-axis about the y-axis.

Hint : Start with sketching the bounded region.

Step 1
We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.

Hint : Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative disk.

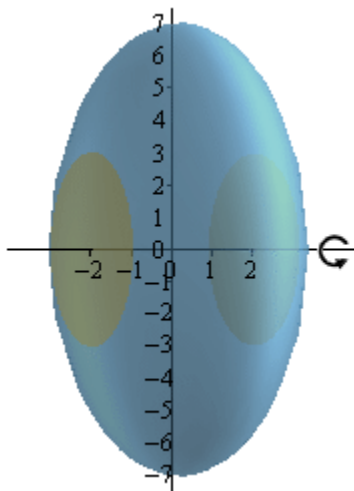Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative disk can be of great help when we go to write down the area formula. Also, getting the representative disk can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2
Here is a sketch of the solid of revolution.

Here are a couple of sketches of a representative disk. The image on the left shows a representative disk with the front half of the solid cut away and the image on the right shows a representative disk with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).

Hint : Determine a formula for the area of the disk.

Step 3
We now need to find a formula for the area of the disk. Because we are using disks that are centered on the *y*-axis we know that the area formula will need to be in terms of *y*. This in turn means that we'll need to rewrite the equation of the boundary curve to get into terms of *y*.

Here is another sketch of a representative disk with all of the various quantities we need put into it.
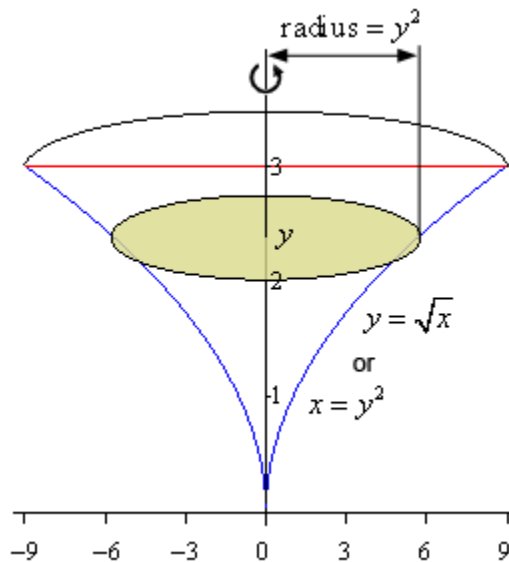
As we can see from the sketch the disk is centered on the *y*-axis and placed at some *y*. The radius of the disk is the distance from the *y*-axis to the curve defining the edge of the solid. In other words,

$$\text{Radius} = y^2$$

The area of the disk is then,

$$A(y) = \pi(\text{Radius})^2 = \pi(y^2)^2 = \pi y^4$$

Step 4
The final step is to then set up the integral for the volume and evaluate it.

For the limits on the integral we can see that the "first" disk in the solid would occur at $y = 0$ and the "last" disk would occur at $y = 3$. Our limits are then : $0 \le y \le 3$.

The volume is then,

$$V = \int_0^3 \pi y^4 \, dy = \tfrac{1}{5}\pi y^5 \Big|_0^3 = \boxed{\tfrac{243}{5}\pi}$$

2. Use the method of disks/rings to determine the volume of the solid obtained by rotating the region bounded by $y = 7 - x^2$, $x = -2$, $x = 2$ and the *x*-axis about the *x*-axis.

Hint : Start with sketching the bounded region.

Step 1
We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.

$$y = 7 - x^2$$

$x = -2$ $x = 2$

Hint : Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative disk.

Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative disk can be of great help when we go to write down the area formula. Also, getting the representative disk can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2
Here is a sketch of the solid of revolution.

Here are a couple of sketches of a representative disk. The image on the left shows a representative disk with the front half of the solid cut away and the image on the right shows a representative disk with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).

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Hint : Determine a formula for the area of the disk.

Step 3

We now need to find a formula for the area of the disk. Because we are using disks that are centered on the *x*-axis we know that the area formula will need to be in terms of *x*. Therefore the equation of the curve will need to be in terms of *x* (which in this case it already is).

Here is another sketch of a representative disk with all of the various quantities we need put into it.
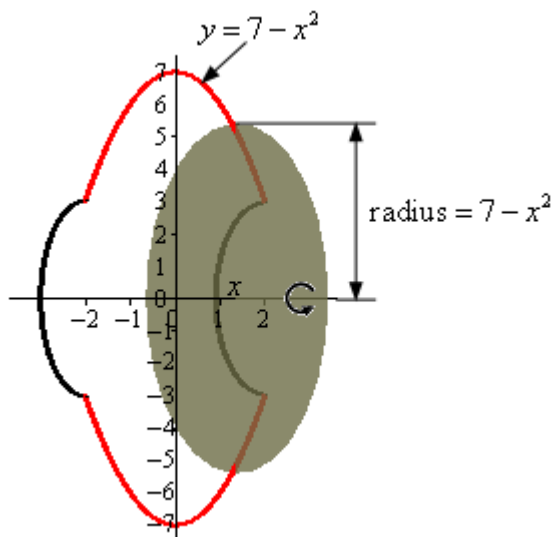
As we can see from the sketch the disk is centered on the *x*-axis and placed at some *x*. The radius of the disk is the distance from the *x*-axis to the curve defining the edge of the solid. In other words,

$$\text{Radius } = 7 - x^2$$

The area of the disk is then,

$$A(x) = \pi (\text{Radius})^2 = \pi (7 - x^2)^2 = \pi (49 - 14x^2 + x^4)$$

Step 4

The final step is to then set up the integral for the volume and evaluate it.

For the limits on the integral we can see that the "first" disk in the solid would occur at $x = -2$ and the "last" disk would occur at $x = 2$. Our limits are then : $-2 \le x \le 2$.

The volume is then,

$$V = \int_{-2}^{2} \pi (49 - 14x^2 + x^4) dx = \pi \left(49x - \tfrac{14}{3}x^3 + \tfrac{1}{5}x^5\right)\Big|_{-2}^{2} = \boxed{\tfrac{2012}{15}\pi}$$

---

3. Use the method of disks/rings to determine the volume of the solid obtained by rotating the region bounded by $x = y^2 - 6y + 10$ and $x = 5$ about the y-axis.

Hint : Start with sketching the bounded region.

Step 1

We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.

Here is the work used to determine the intersection points (we'll need these later).

$$y^2 - 6y + 10 = 5$$
$$y^2 - 6y + 5 = 0$$
$$(y-5)(y-1) = 0 \qquad \Rightarrow \qquad y = 1, \quad y = 5 \qquad \Rightarrow \qquad (5,1) \ \& \ (5,5)$$

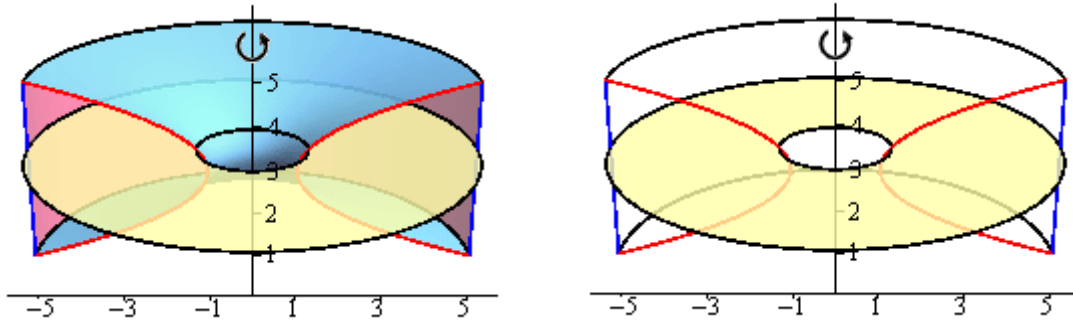Hint : Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative ring.

Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative ring can be of great help when we go to write down the area formula. Also, getting the representative ring can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2
Here is a sketch of the solid of revolution.

Here are a couple of sketches of a representative ring. The image on the left shows a representative ring with the front half of the solid cut away and the image on the right shows a representative ring with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).

Hint : Determine a formula for the area of the ring.

Step 3
We now need to find a formula for the area of the ring. Because we are using rings that are centered on the *y*-axis we know that the area formula will need to be in terms of *y*. Therefore the equation of the curves will need to be in terms of *y* (which in this case they already are).

Here is another sketch of a representative ring with all of the various quantities we need put into it.

As we can see from the sketch the ring is centered on the *y*-axis and placed at some *y*. The inner radius of the ring is the distance from the *y*-axis to the curve defining the inner edge of the solid. The outer radius of the ring is the distance from the *y*-axis to the curve defining the outer edge of the solid. In other words,

$$\text{Inner Radius} = y^2 - 6y + 10 \qquad\qquad \text{Outer Radius} = 5$$

The area of the ring is then,

$$A(x) = \pi \left[(\text{Outer Radius})^2 - (\text{Inner Radius})^2 \right]$$
$$= \pi \left[(5)^2 - (y^2 - 6y + 10)^2 \right] = \pi \left(-75 + 120y - 56y^2 + 12y^3 - y^4 \right)$$

Step 4
The final step is to then set up the integral for the volume and evaluate it.

From the intersection points shown in the graph from Step 1 we can see that the "first" ring in the solid would occur at $y = 1$ and the "last" ring would occur at $y = 5$. Our limits are then : $1 \le y \le 5$.

The volume is then,

$$V = \int_1^5 \pi \left(-75 + 120y - 56y^2 + 12y^3 - y^4 \right) dy$$
$$= \pi \left(-75y + 60y^2 - \tfrac{56}{3} y^3 + 3y^4 - \tfrac{1}{5} y^5 \right) \Big|_1^5 = \boxed{\tfrac{1088}{15} \pi}$$

---

4. Use the method of disks/rings to determine the volume of the solid obtained by rotating the region bounded by $y = 2x^2$ and $y = x^3$ about the *x*-axis.

Hint : Start with sketching the bounded region.

Step 1
We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.

Here is the work used to determine the intersection points (we'll need these later).
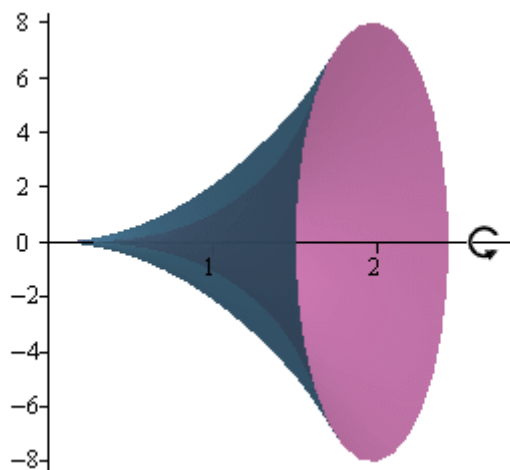
$$x^3 = 2x^2$$
$$x^3 - 2x^2 = 0$$
$$x^2(x-2) = 0 \quad \Rightarrow \quad x = 0, \quad x = 2 \quad \Rightarrow \quad (0,0) \ \& \ (2,8)$$

Hint : Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative ring.

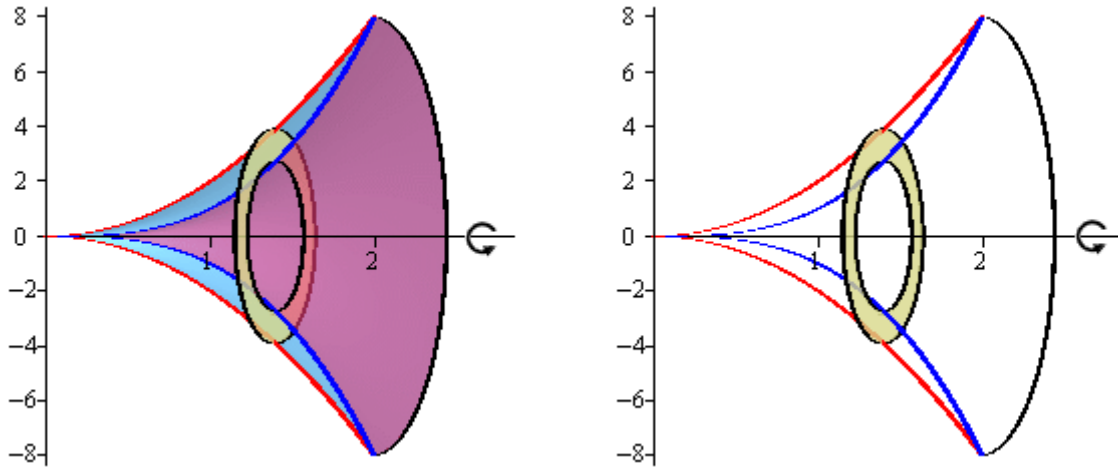Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative ring can be of great help when we go to write down the area formula. Also, getting the representative ring can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2
Here is a sketch of the solid of revolution.

Here are a couple of sketches of a representative ring. The image on the left shows a representative ring with the front half of the solid cut away and the image on the right shows a representative ring with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).

Hint : Determine a formula for the area of the ring.

Step 3

We now need to find a formula for the area of the ring. Because we are using rings that are centered on the *x*-axis we know that the area formula will need to be in terms of *x*. Therefore the equation of the curves will need to be in terms of *x* (which in this case they already are).

Here is another sketch of a representative ring with all of the various quantities we need put into it.

As we can see from the sketch the ring is centered on the *x*-axis and placed at some *x*. The inner radius of the ring is the distance from the *x*-axis to the curve defining the inner edge of the solid.

The outer radius of the ring is the distance from the *x*-axis to the curve defining the outer edge of the solid. In other words,

$$\text{Inner Radius} = x^3 \qquad\qquad \text{Outer Radius} = 2x^2$$

The area of the ring is then,

$$A(x) = \pi\left[\left(\text{Outer Radius}\right)^2 - \left(\text{Inner Radius}\right)^2\right]$$
$$= \pi\left[\left(2x^2\right)^2 - \left(x^3\right)^2\right] = \pi\left(4x^4 - x^6\right)$$

Step 4
The final step is to then set up the integral for the volume and evaluate it.

From the intersection points shown in the graph from Step 1 we can see that the "first" ring in the solid would occur at $x=0$ and the "last" ring would occur at $x=2$. Our limits are then : $0 \le x \le 2$.

The volume is then,

$$V = \int_0^2 \pi\left(4x^4 - x^6\right)dx = \pi\left(\tfrac{4}{5}x^5 - \tfrac{1}{7}x^7\right)\Big|_0^2 = \boxed{\tfrac{256}{35}\pi}$$

---

5. Use the method of disks/rings to determine the volume of the solid obtained by rotating the region bounded by $y = 6e^{-2x}$ and $y = 6 + 4x - 2x^2$ between $x = 0$ and $x = 1$ about the line $y = -2$.

Hint : Start with sketching the bounded region.

Step 1
We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

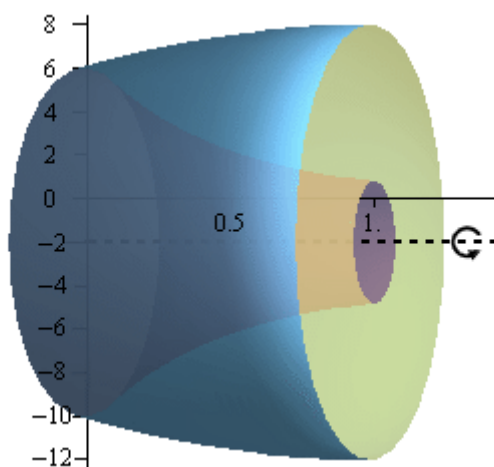Here is a sketch of the bounded region with the axis of rotation shown.
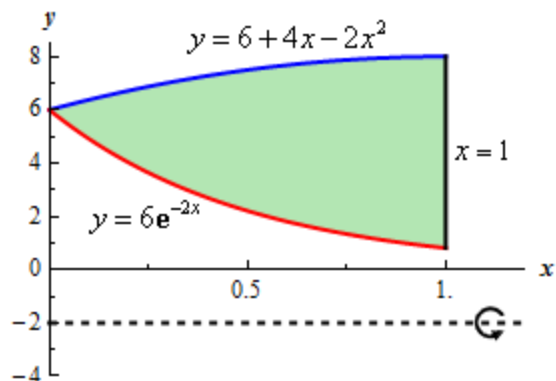
For the intersection point on the left a quick check by plugging $x = 0$ into both equations shows that the intersection point is in fact $(0, 6)$ as we might have guessed from the graph. We'll be needing this point in a bit. From the sketch of the region it is also clear that there is no intersection point on the right.

Hint : Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative ring.

Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative ring can be of great help when we go to write down the area formula. Also, getting the representative ring can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2
Here is a sketch of the solid of revolution.

Here are a couple of sketches of a representative ring. The image on the left shows a representative ring with the front half of the solid cut away and the image on the right shows a

representative ring with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).
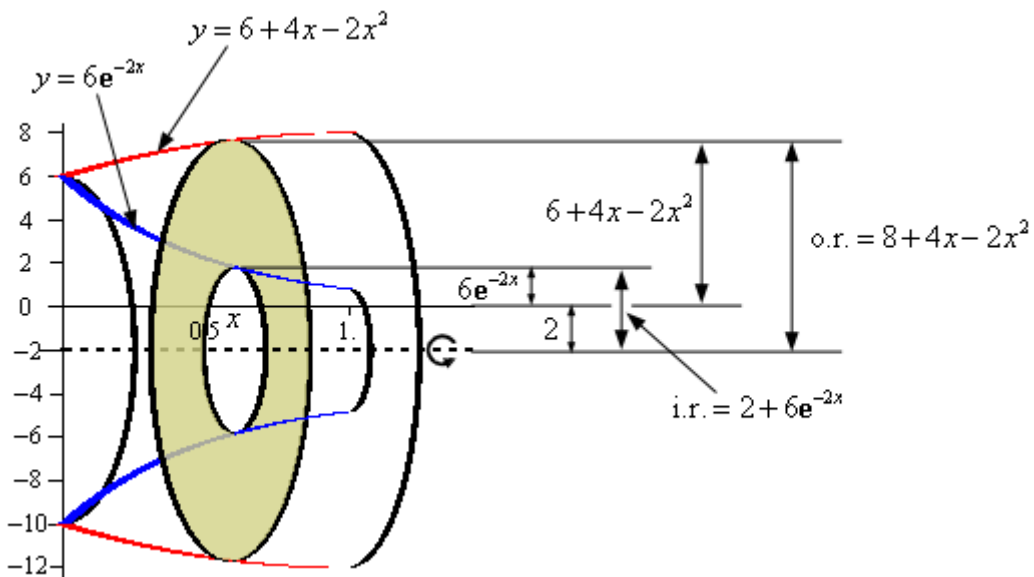

Hint : Determine a formula for the area of the ring.

Step 3
We now need to find a formula for the area of the ring. Because we are using rings that are centered on a horizontal axis (*i.e.* parallel to the *x*-axis) we know that the area formula will need to be in terms of *x*. Therefore the equations of the curves will need to be in terms of *x* (which in this case they already are).

Here is another sketch of a representative ring with all of the various quantities we need put into it.

From the sketch we can see the ring is centered on the line $y = -2$ and placed at some x.

The inner radius of the ring is the distance from the axis of rotation to the x-axis (a distance of 2) followed by the distance from the x-axis to the curve defining the inner edge of the solid (a distance of $6\mathbf{e}^{-2x}$).

Likewise, the outer radius of the ring is the distance from the axis of rotation to the x-axis (again, a distance of 2) followed by the distance from the x-axis to the curve defining the outer edge of the solid (a distance of $6 + 4x - 2x^2$).

So the inner and outer radii are,

$$\text{Inner Radius} = 2 + 6\mathbf{e}^{-2x} \qquad \text{Outer Radius} = 2 + 6 + 4x - 2x^2 = 8 + 4x - 2x^2$$

The area of the ring is then,

$$
\begin{aligned}
A(x) &= \pi\left[\left(\text{Outer Radius}\right)^2 - \left(\text{Inner Radius}\right)^2\right] \\
&= \pi\left[\left(8 + 4x - 2x^2\right)^2 - \left(2 + 6\mathbf{e}^{-2x}\right)^2\right] \\
&= \pi\left(60 + 64x - 16x^2 - 16x^3 + 4x^4 - 24\mathbf{e}^{-2x} - 36\mathbf{e}^{-4x}\right)
\end{aligned}
$$

Step 4
The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" ring in the solid would occur at $x = 0$ and the "last" ring would occur at $x = 1$. Our limits are then : $0 \le x \le 1$.

The volume is then,

$$
\begin{aligned}
V &= \int_0^1 \pi\left(60 + 64x - 16x^2 - 16x^3 + 4x^4 - 24\mathbf{e}^{-2x} - 36\mathbf{e}^{-4x}\right) dx \\
&= \pi\left(60x + 32x^2 - \tfrac{16}{3}x^3 - 4x^4 + \tfrac{4}{5}x^5 + 12\mathbf{e}^{-2x} + 9\mathbf{e}^{-4x}\right)\Big|_0^1 = \boxed{\left(\tfrac{937}{15} + 12\mathbf{e}^{-2} + 9\mathbf{e}^{-4}\right)\pi}
\end{aligned}
$$

---

6. Use the method of disks/rings to determine the volume of the solid obtained by rotating the region bounded by $y = 10 - 6x + x^2$, $y = -10 + 6x - x^2$, $x = 1$ and $x = 5$ about the line $y = 8$.
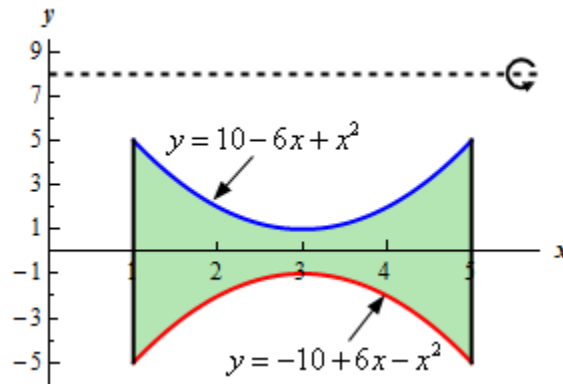
Hint : Start with sketching the bounded region.

Step 1
We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.
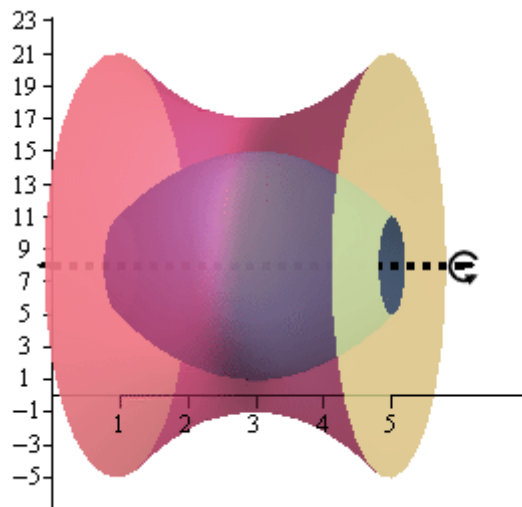

Hint : Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative ring.

Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative ring can be of great help when we go to write down the area formula. Also, getting the representative ring can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2
Here is a sketch of the solid of revolution.

Here are a couple of sketches of a representative ring. The image on the left shows a representative ring with the front half of the solid cut away and the image on the right shows a representative ring with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).
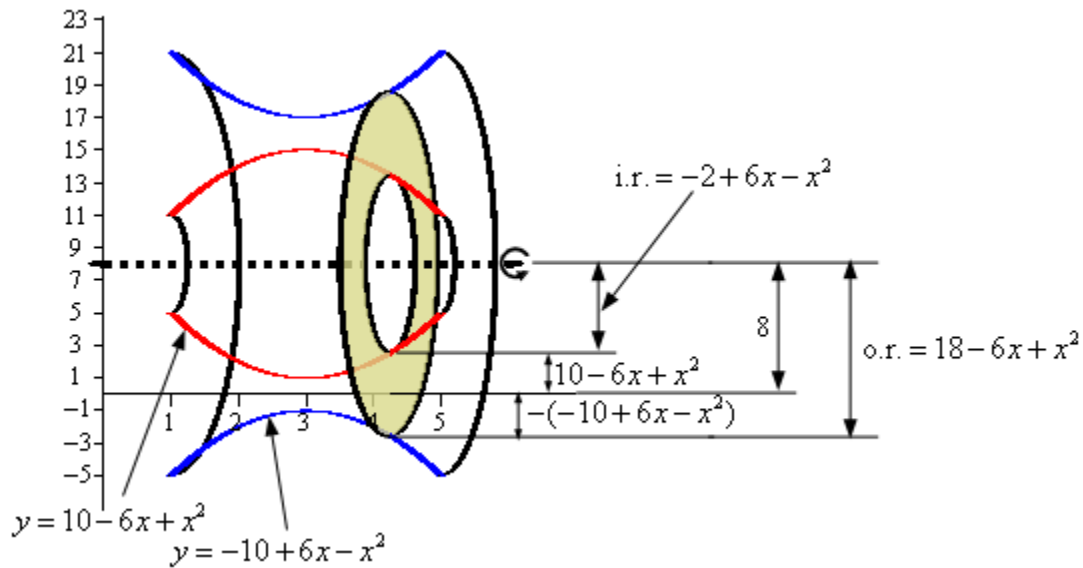

Hint : Determine a formula for the area of the ring.

Step 3

We now need to find a formula for the area of the ring. Because we are using rings that are centered on a horizontal axis (*i.e.* parallel to the *x*-axis) we know that the area formula will need to be in terms of *x*. Therefore the equations of the curves will need to be in terms of *x* (which in this case they already are).

Here is another sketch of a representative ring with all of the various quantities we need put into it.

From the sketch we can see the ring is centered on the line $y = 8$ and placed at some x.

The inner radius of the ring is then the distance from the axis of rotation to the curve defining the inner edge of the solid. To determine a formula for this first notice that the axis of rotation is a distance of 8 from the x-axis. Next, the curve defining the inner edge of the solid is a distance of $y = 10 - 6x + x^2$ from the x-axis. The inner radius is then the difference between these two distances or,

$$\text{Inner Radius} = 8 - \left(10 - 6x + x^2\right) = -2 + 6x - x^2$$

The outer radius is computed in a similar manner. It is the distance from the axis of rotation to the x-axis (a distance of 8) and then it continues below the x-axis until it reaches the curve defining the outer edge of the solid. So, we need to add these two distances but we need to be careful because the "lower" function is in fact negative value and so the distance of the point on the lower function from the x-axis is in fact : $-\left(-10 + 6x - x^2\right)$ as is shown on the sketch. The negative in front of the equation makes sure that the negative value of the function is turned into a positive quantity (which we need for our distance). The outer radius is then the sum of these two distances or,

$$\text{Outer Radius} = 8 - \left(-10 + 6x - x^2\right) = 18 - 6x + x^2$$

The area of the ring is then,

$$A(x) = \pi \left[\left(\text{Outer Radius} \right)^2 - \left(\text{Inner Radius} \right)^2 \right]$$
$$= \pi \left[\left(18 - 6x + x^2 \right)^2 - \left(-2 + 6x - x^2 \right)^2 \right] = \pi \left(320 - 192x + 32x^2 \right)$$

Step 4
The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" ring in the solid would occur at $x = 1$ and the "last" ring would occur at $x = 5$. Our limits are then : $1 \le x \le 5$.

The volume is then,

$$V = \int_1^5 \pi \left(320 - 192x + 32x^2 \right) dx = \pi \left(320x - 96x^2 + \tfrac{32}{3} x^3 \right) \Big|_1^5 = \boxed{\tfrac{896}{3} \pi}$$

---

7. Use the method of disks/rings to determine the volume of the solid obtained by rotating the region bounded by $x = y^2 - 4$ and $x = 6 - 3y$ about the line $x = 24$.

Hint : Start with sketching the bounded region.

Step 1
We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.
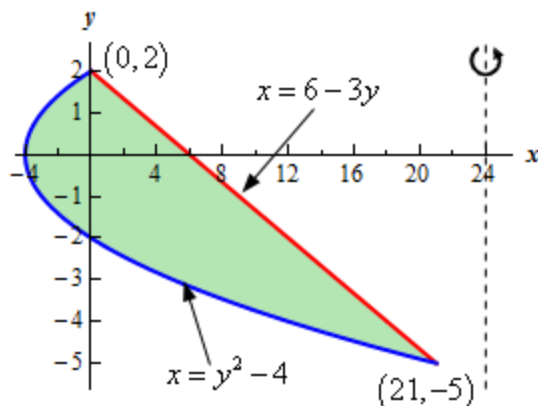
To get the intersection points shown on the sketch all we need to do is set the two equations equal and solve (we'll need these in a bit).
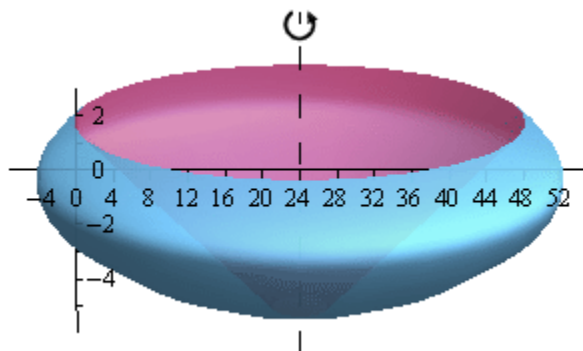
$$y^2 - 4 = 6 - 3y$$
$$y^2 + 3y - 10 = 0$$
$$(y+5)(y-2) = 0 \quad \Rightarrow \quad y = -5, \ \ y = 2 \quad \Rightarrow \quad (-5, 21) \ \& \ (2, 0)$$

Hint : Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative ring.

Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative ring can be of great help when we go to write down the area formula. Also, getting the representative ring can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2
Here is a sketch of the solid of revolution.

Here are a couple of sketches of a representative ring. The image on the left shows a representative ring with the front half of the solid cut away and the image on the right shows a

representative ring with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).
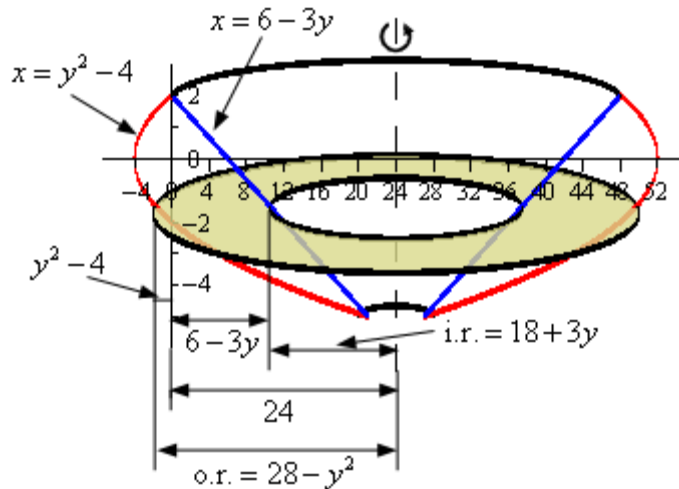

Hint : Determine a formula for the area of the ring.

Step 3
We now need to find a formula for the area of the ring. Because we are using rings that are centered on a vertical axis (*i.e.* parallel to the *y*-axis) we know that the area formula will need to be in terms of *y*. Therefore the equation of the curves will need to be in terms of *y* (which in this case they already are).

Here is another sketch of a representative ring with all of the various quantities we need put into it.

From the sketch we can see the ring is centered on the line $x = 24$ and placed at some *y*.

The inner radius of the ring is then the distance from the axis of rotation to the curve defining the inner edge of the solid. To determine a formula for this first notice that the axis of rotation is a distance of 24 from the *y*-axis. Next, the curve defining the inner edge of the solid is a distance of

$x = 6 - 3y$ from the *y*-axis. The inner radius is then the difference between these two distances or,

$$\text{Inner Radius} = 24 - (6 - 3y) = 18 + 3y$$

The outer radius is computed in a similar manner but is a little trickier. In this case the curve defining the outer edge of the solid occurs on both the left and right of the *y*-axis.

Let's first look at the case as shown in the sketch above. In this case the value of the function defining the outer edge of the solid is to the left of the *y*-axis and so has a negative value. The distance of this point from the *y*-axis is then $-(y^2 - 4)$ where the minus sign turns the negative function value into a positive value that we need for distance. The outer radius for this case is then the sum of the distance of the axis of rotation to the *y*-axis (a distance of 24) and the distance of the curve defining the outer edge to the *y*-axis (which we found above).

If the curve defining the outer edge of the solid is to the right of the *y*-axis then it will have a positive value and so the distance of points on the curve and the *y*-axis is just $y^2 - 4$. We don't need the minus sign in this case because the function value is already positive, which we need for distance. The outer radius in this case is then the distance from the axis of rotation to the *y*-axis (a distance of 24) minus this new distance.

Nicely enough in either case the outer radius is then,

$$\text{Outer Radius} = 24 - (y^2 - 4) = 28 - y^2$$

Note that in cases like this where the curve defining an edge has both positive and negative values the final equation of the radius (inner or outer depending on the problem) will be the same. You just need to be careful in setting up the case you choose to look at. If you get the first case set up correctly you won't need to do the second as the formula will be the same.

The area of the ring is then,

$$A(x) = \pi \left[(\text{Outer Radius})^2 - (\text{Inner Radius})^2 \right]$$
$$= \pi \left[(28 - y^2)^2 - (18 + 3y)^2 \right] = \pi \left(460 - 108y - 65y^2 + y^4 \right)$$

Step 4
The final step is to then set up the integral for the volume and evaluate it.

From the intersection points of the two curves we found in Step 1 we can see that the "first" ring in the solid would occur at $y = -5$ and the "last" ring would occur at $y = 2$. Our limits are then : $-5 \le y \le 2$.

The volume is then,

$$V = \int_{-5}^{2} \pi \left(460 - 108y - 65y^2 + y^4 \right) dy = \pi \left(460y - 54y^2 - \tfrac{65}{3} y^3 + \tfrac{1}{5} y^5 \right) \Big|_{-5}^{2} = \boxed{\tfrac{31556}{15} \pi}$$

8. Use the method of disks/rings to determine the volume of the solid obtained by rotating the region bounded by $y = 2x + 1$, $x = 4$ and $y = 3$ about the line $x = -4$.

Hint : Start with sketching the bounded region.

Step 1
We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.

Hint : Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative ring.
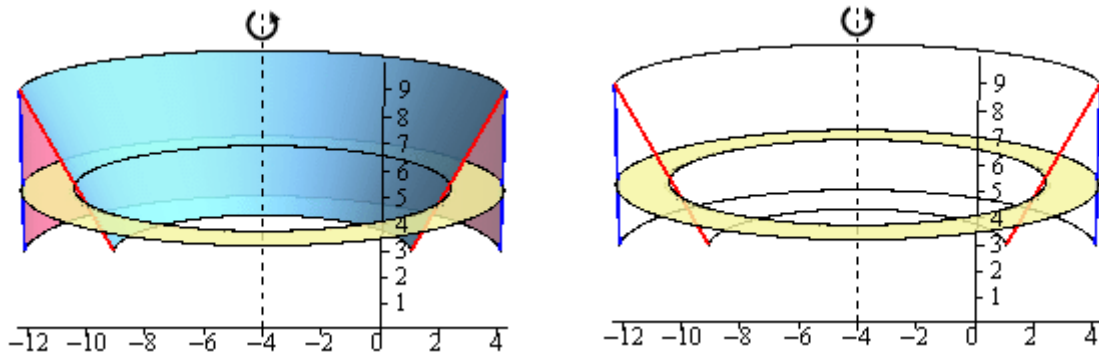
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative ring can be of great help when we go to write down the area formula. Also, getting the representative ring can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2
Here is a sketch of the solid of revolution.

Here are a couple of sketches of a representative ring. The image on the left shows a representative ring with the front half of the solid cut away and the image on the right shows a representative ring with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).


Hint : Determine a formula for the area of the ring.

Step 3
We now need to find a formula for the area of the ring. Because we are using rings that are centered on a vertical axis (*i.e.* parallel to the *y*-axis) we know that the area formula will need to be in terms of *y*. Therefore the equations of the curves will need to be in terms of *y* and so we'll need to rewrite the equation of the line to be in terms of *y*.

Here is another sketch of a representative ring with all of the various quantities we need put into it.
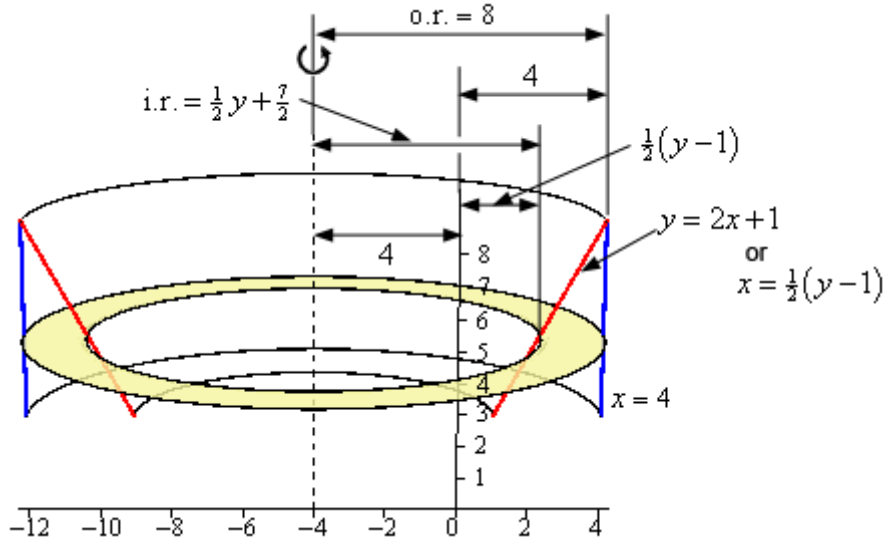
From the sketch we can see the ring is centered on the line $x = -4$ and placed at some y.

The inner radius of the ring is the distance from the axis of rotation to the y-axis (a distance of 4) followed by the distance from the y-axis to the curve defining the inner edge of the solid (a distance of $\frac{1}{2}(y-1)$).

Likewise, the outer radius of the ring is the distance from the axis of rotation to the y-axis (again, a distance of 4) followed by the distance from the y-axis to the curve defining the outer edge of the solid (a distance of 4).

So the inner and outer radii are,

$$\text{Inner Radius} = 4 + \tfrac{1}{2}(y-1) = \tfrac{1}{2}y + \tfrac{7}{2} \qquad\qquad \text{Outer Radius} = 4 + 4 = 8$$

The area of the ring is then,

$$A(x) = \pi\left[\left(\text{Outer Radius}\right)^2 - \left(\text{Inner Radius}\right)^2\right]$$
$$= \pi\left[(8)^2 - \left(\tfrac{1}{2}y + \tfrac{7}{2}\right)^2\right] = \pi\left(\tfrac{207}{4} - \tfrac{7}{2}y - \tfrac{1}{4}y^2\right)$$

Step 4
The final step is to then set up the integral for the volume and evaluate it.

From the intersection points of the two curves we found in Step 1 we can see that the "first" ring in the solid would occur at $y = 3$ and the "last" ring would occur at $y = 9$. Our limits are then : $3 \le y \le 9$.

The volume is then,

$$V = \int_3^9 \pi \left(\tfrac{207}{4} - \tfrac{7}{2}y - \tfrac{1}{4}y^2 \right) dx = \pi \left(\tfrac{207}{4}y - \tfrac{7}{4}y^2 - \tfrac{1}{12}y^3 \right)\Big|_3^9 = \boxed{126\pi}$$

Volumes of Solids of Revolution / Method of Cylinders

1. Use the method of cylinders to determine the volume of the solid obtained by rotating the region bounded by $x = (y-2)^2$, the *x*-axis and the *y*-axis about the *x*-axis.

Hint : Start with sketching the bounded region.

Step 1
We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.
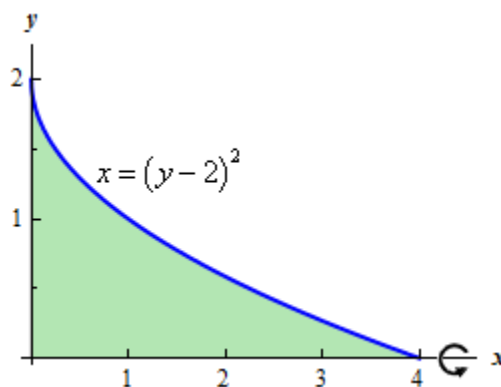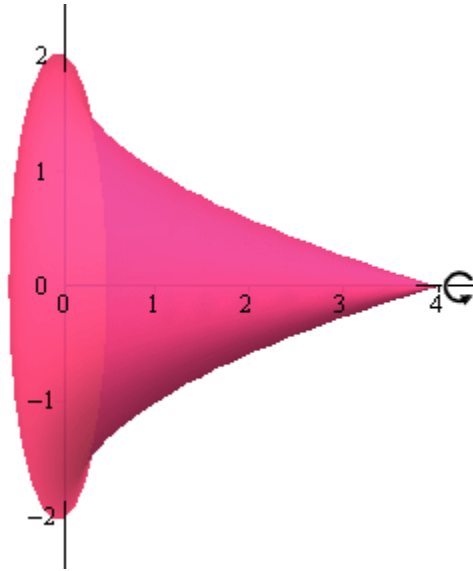

Note that we only used the lower half of the parabola here because if we also included the upper half there would be nothing to bound the region above it. Therefore, in order for the *x*-axis and *y*-axis to be bounding curves we have to use the portion below the lower half of the parabola.

Hint : Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative cylinder.
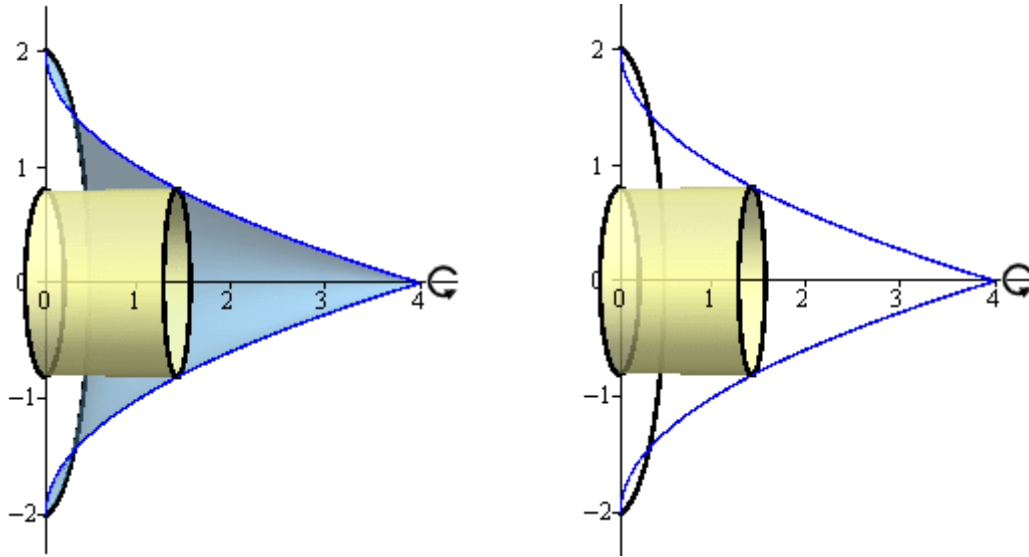
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative cylinder can be of great help when we go to write down the area formula. Also, getting the representative cylinder can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2
Here is a sketch of the solid of revolution.

Here are a couple of sketches of a representative cylinder. The image on the left shows a representative cylinder with the front half of the solid cut away and the image on the right shows a representative cylinder with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).
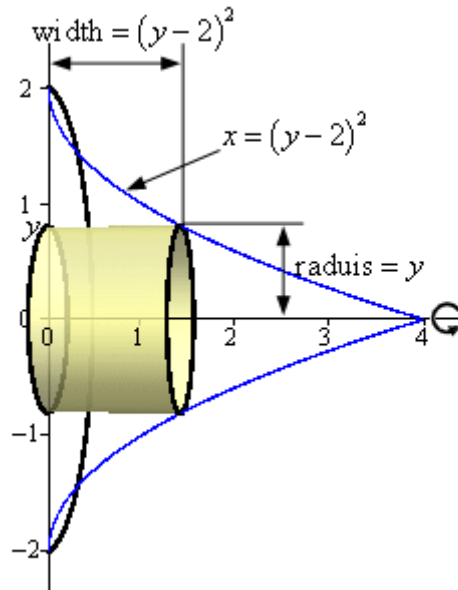
Hint : Determine a formula for the area of the cylinder.

Step 3
We now need to find a formula for the area of the cylinder. Because we are using cylinders that are centered on the x-axis we know that the area formula will need to be in terms of y. Therefore the equation of the curves will need to be in terms of y (which in this case they already are).

Here is another sketch of a representative cylinder with all of the various quantities we need put into it.

From the sketch we can see the cylinder is centered on the x-axis and the upper edge of the cylinder is at some y.

The radius of the cylinder is just the distance from the x-axis to the upper edge of the cylinder (*i.e.* y). The width of the cylinder is the distance from the y-axis to the curve defining the edge of the solid (a distance of $(y-2)^2$).

So the radius and width of the cylinder are,

$$\text{Radius} = y \qquad\qquad \text{Width} = (y-2)^2$$

The area of the cylinder is then,

$$A(y) = 2\pi(\text{Radius})(\text{Width}) = 2\pi(y)(y-2)^2 = 2\pi\left(4y - 4y^2 + y^3\right)$$

Step 4
The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" cylinder in the solid would occur at $y = 0$ and the "last" cylinder would occur at $y = 2$. Our limits are then : $0 \le y \le 2$.

The volume is then,

$$V = \int_0^2 2\pi\left(4y - 4y^2 + y^3\right)dy = 2\pi\left(2y^2 - \tfrac{4}{3}y^3 + \tfrac{1}{4}y^4\right)\Big|_0^2 = \boxed{\tfrac{8}{3}\pi}$$

---

2. Use the method of cylinders to determine the volume of the solid obtained by rotating the region bounded by $y = \dfrac{1}{x}$, $x = \dfrac{1}{2}$, $x = 4$ and the x-axis about the y-axis.
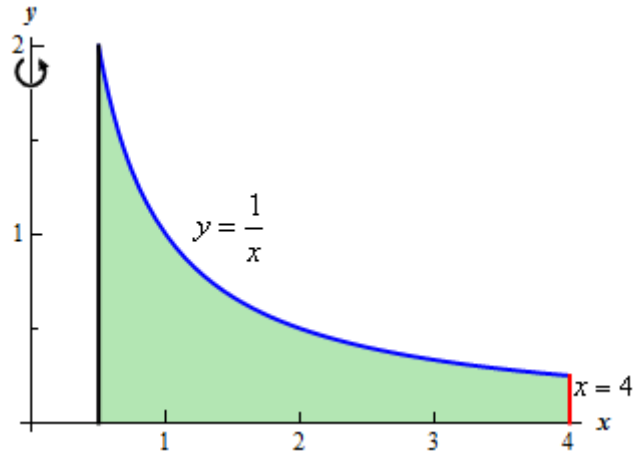
Hint : Start with sketching the bounded region.

Step 1
We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

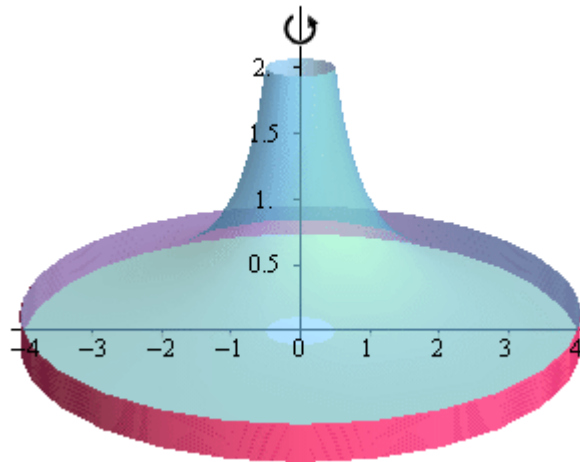Here is a sketch of the bounded region with the axis of rotation shown.

Hint : Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative cylinder.
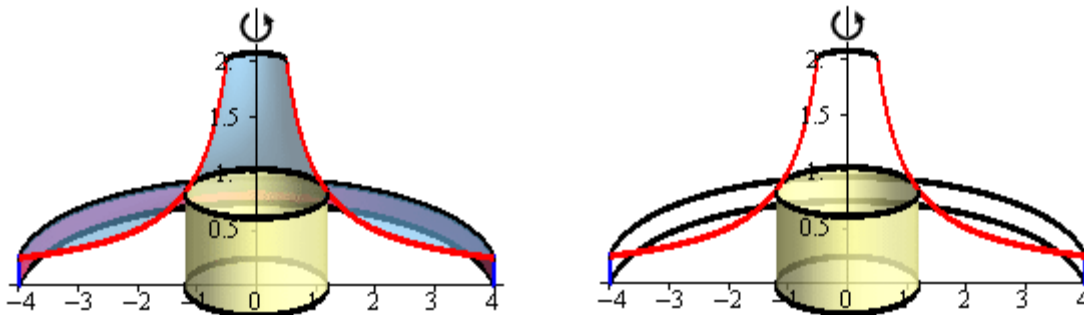
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative cylinder can be of great help when we go to write down the area formula. Also, getting the representative cylinder can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2
Here is a sketch of the solid of revolution.

Here are a couple of sketches of a representative cylinder. The image on the left shows a representative cylinder with the front half of the solid cut away and the image on the right shows a representative cylinder with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).
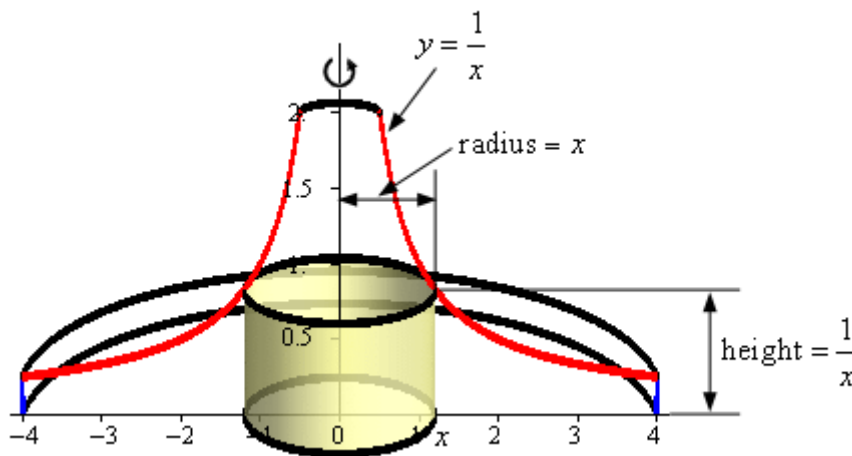
Hint : Determine a formula for the area of the cylinder.

Step 3

We now need to find a formula for the area of the cylinder. Because we are using cylinders that are centered on the *y*-axis we know that the area formula will need to be in terms of *x*. Therefore the equation of the curves will need to be in terms of *x* (which in this case they already are).

Here is another sketch of a representative cylinder with all of the various quantities we need put into it.

From the sketch we can see the cylinder is centered on the *y*-axis and the right edge of the cylinder is at some *x*.

The radius of the cylinder is just the distance from the *y*-axis to the right edge of the cylinder (*i.e.* *x*). The height of the cylinder is the distance from the *x*-axis to the curve defining the edge of the solid (a distance of $\frac{1}{x}$).

So the radius and width of the cylinder are,

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$$\text{Radius} = x \qquad\qquad \text{Height} = \frac{1}{x}$$

The area of the cylinder is then,

$$A(x) = 2\pi\,(\text{Radius})(\text{Height}) = 2\pi\,(x)\left(\frac{1}{x}\right) = 2\pi$$

Do not expect all the variables to cancel out in the area formula. It may happen on occasion, as it did here, but it is rare with it does.

Step 4
The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" cylinder in the solid would occur at $x = \frac{1}{2}$ and the "last" cylinder would occur at $x = 4$. Our limits are then : $\frac{1}{2} \le x \le 4$.

The volume is then,

$$V = \int_{\frac{1}{2}}^{4} 2\pi\,dx = 2\,\pi\,(x)\Big|_{\frac{1}{2}}^{4} = \boxed{7\pi}$$

---

3. Use the method of cylinders to determine the volume of the solid obtained by rotating the region bounded by $y = 4x$ and $y = x^3$ about the y-axis. For this problem assume that $x \ge 0$.
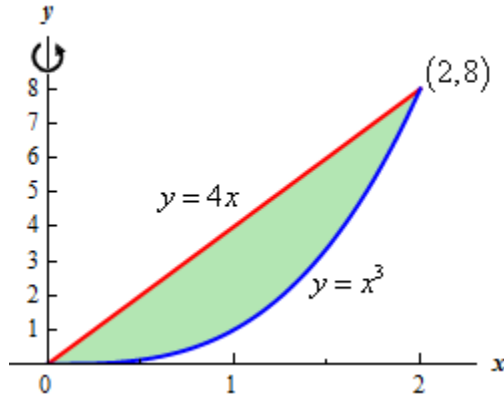
Hint : Start with sketching the bounded region.

Step 1
We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.

To get the intersection points shown on the graph, which we'll need in a bit, all we need to do is set the equations equal to each other and solve.

$$x^3 = 4x$$
$$x^3 - 4x = 0$$
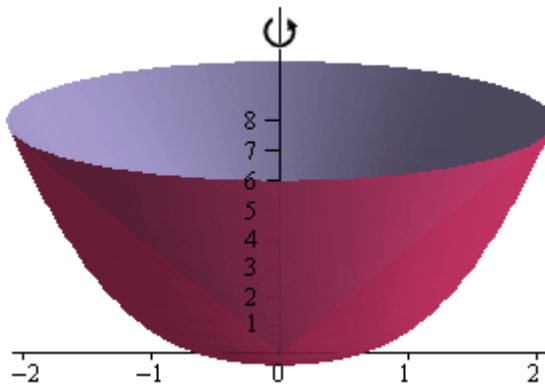$$x(x^2 - 4) = 0 \quad \Rightarrow \quad x = 0, \quad x = \pm 2 \quad \Rightarrow \quad (0,0) \ \& \ (2,8)$$

Note that the problem statement said to assume that $x \geq 0$ and so we won't use the $x = -2$ intersection point.

Hint : Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative cylinder.
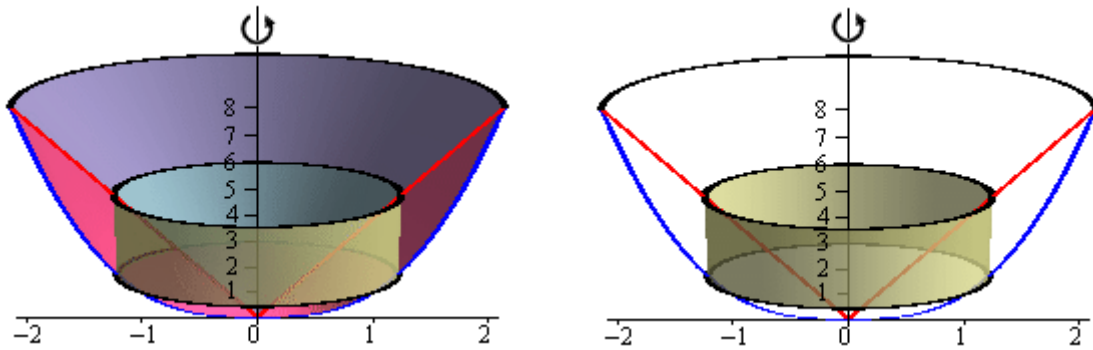
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative cylinder can be of great help when we go to write down the area formula. Also, getting the representative cylinder can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2
Here is a sketch of the solid of revolution.

Here are a couple of sketches of a representative cylinder. The image on the left shows a representative cylinder with the front half of the solid cut away and the image on the right shows a representative cylinder with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).
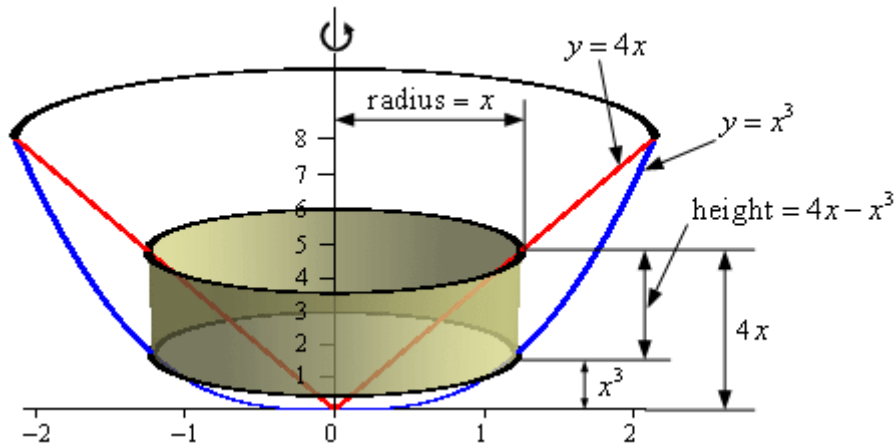

Hint : Determine a formula for the area of the cylinder.

Step 3
We now need to find a formula for the area of the cylinder. Because we are using cylinders that are centered on the *y*-axis we know that the area formula will need to be in terms of *x*. Therefore the equation of the curves will need to be in terms of *x* (which in this case they already are).

Here is another sketch of a representative cylinder with all of the various quantities we need put into it.

From the sketch we can see the cylinder is centered on the *y*-axis and the right edge of the cylinder is at some *x*.

The radius of the cylinder is just the distance from the *y*-axis to the right edge of the cylinder (*i.e. x*).

The top of the cylinder is on the curve defining the upper portion of the solid and is a distance of $4x$ from the *x*-axis. The bottom of the cylinder is on the curve defining the lower portion of the solid and is a distance of x^3 from the *x*-axis. The height then is the difference of these two.

So the radius and height of the cylinder are,

$$\text{Radius} = x \qquad\qquad \text{Height} = 4x - x^3$$

The area of the cylinder is then,

$$A(x) = 2\pi\,(\text{Radius})(\text{Height}) = 2\pi(x)\left(4x - x^3\right) = 2\pi\left(4x^2 - x^4\right)$$

Step 4
The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" cylinder in the solid would occur at $x = 0$ and the "last" cylinder would occur at $x = 2$. Our limits are then : $0 \le x \le 2$.

The volume is then,

$$V = \int_0^2 2\pi\left(4x^2 - x^4\right)dx = 2\pi\left(\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5\right)\Big|_0^2 = \boxed{\tfrac{128}{15}\pi}$$

---

4. Use the method of cylinders to determine the volume of the solid obtained by rotating the region bounded by $y = 4x$ and $y = x^3$ about the *x*-axis. For this problem assume that $x \ge 0$.
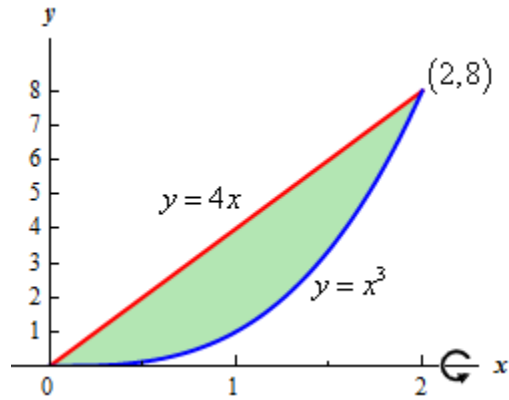
Hint : Start with sketching the bounded region.

Step 1
We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.

To get the intersection points shown on the graph, which we'll need in a bit, all we need to do is set the equations equal to each other and solve.

$$x^3 = 4x$$
$$x^3 - 4x = 0$$
$$x\left(x^2 - 4\right) = 0 \qquad \Rightarrow \qquad x = 0, \quad x = \pm 2 \qquad \Rightarrow \qquad (0,0) \ \& \ (2,8)$$
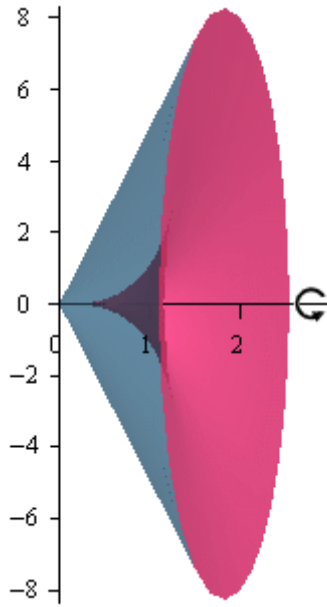
Note that the problem statement said to assume that $x \geq 0$ and so we won't use the $x = -2$ intersection point.

Hint : Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative cylinder.
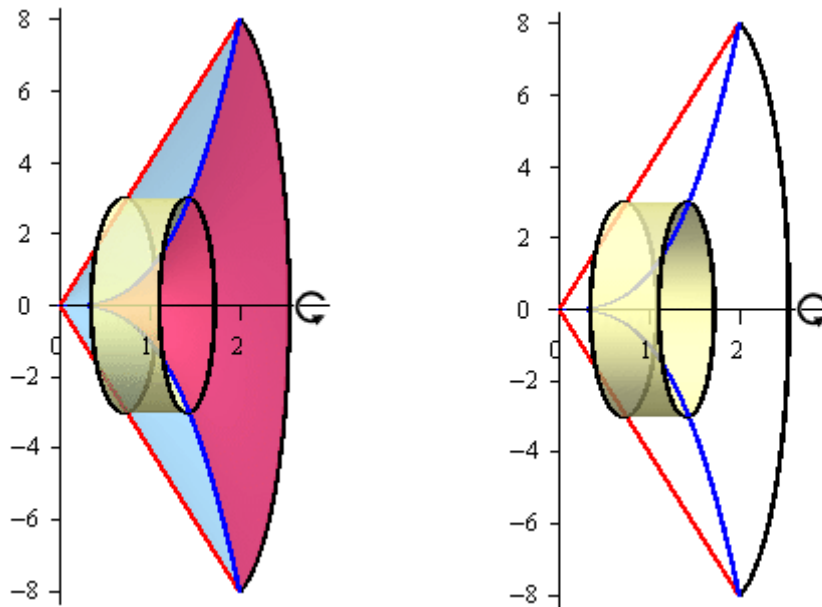
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative cylinder can be of great help when we go to write down the area formula. Also, getting the representative cylinder can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2
Here is a sketch of the solid of revolution.

Here are a couple of sketches of a representative cylinder. The image on the left shows a representative cylinder with the front half of the solid cut away and the image on the right shows a representative cylinder with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).
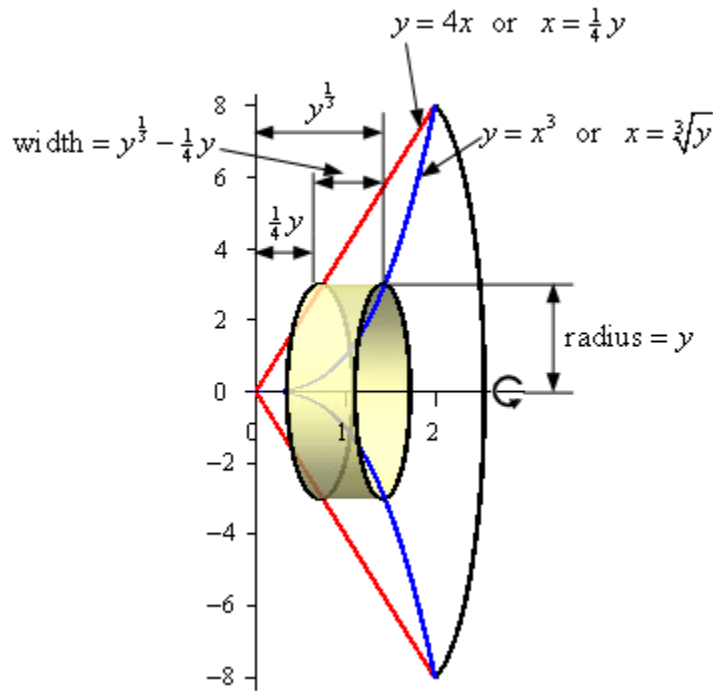

Hint : Determine a formula for the area of the cylinder.

Step 3

We now need to find a formula for the area of the cylinder. Because we are using cylinders that are centered on the *x*-axis we know that the area formula will need to be in terms of *y*. Therefore we'll need to rewrite the equations of the curves in terms of *y*.

Here is another sketch of a representative cylinder with all of the various quantities we need put into it.

From the sketch we can see the cylinder is centered on the *x*-axis and the upper edge of the cylinder is at some *y*.

The radius of the cylinder is just the distance from the *x*-axis to the upper edge of the cylinder (*i.e. y*).

The right edge of the cylinder is on the curve defining the right portion of the solid and is a distance of $y^{\frac{1}{3}}$ from the *y*-axis. The left edge of the cylinder is on the curve defining the left portion of the solid and is a distance of $\frac{1}{4}y$ from the *y*-axis. The height then is the difference of these two.

So the radius and width of the cylinder are,

$$\text{Radius} = y \qquad\qquad \text{Width} = y^{\frac{1}{3}} - \tfrac{1}{4}y$$

The area of the cylinder is then,

$$A(y) = 2\pi (\text{Radius})(\text{Height}) = 2\pi (y)\left(y^{\frac{1}{3}} - \tfrac{1}{4}y\right) = 2\pi \left(y^{\frac{4}{3}} - \tfrac{1}{4}y^2\right)$$

Step 4
The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" cylinder in the solid would occur at $y = 0$ and the "last" cylinder would occur at $y = 8$. Our limits are then : $0 \le y \le 8$.

The volume is then,

$$V = \int_0^8 2\pi \left(y^{\frac{4}{3}} - \tfrac{1}{4}y^2\right) dy = 2\pi \left(\tfrac{3}{7}y^{\frac{7}{3}} - \tfrac{1}{12}y^3\right)\Big|_0^8 = \boxed{\tfrac{512}{21}\pi}$$

5. Use the method of cylinders to determine the volume of the solid obtained by rotating the region bounded by $y = 2x+1$, $y = 3$ and $x = 4$ about the line $y = 10$.
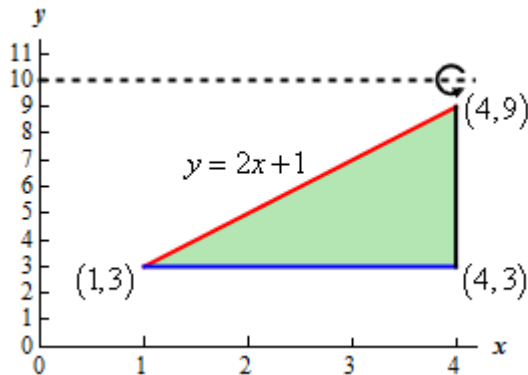
Hint : Start with sketching the bounded region.

Step 1
We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

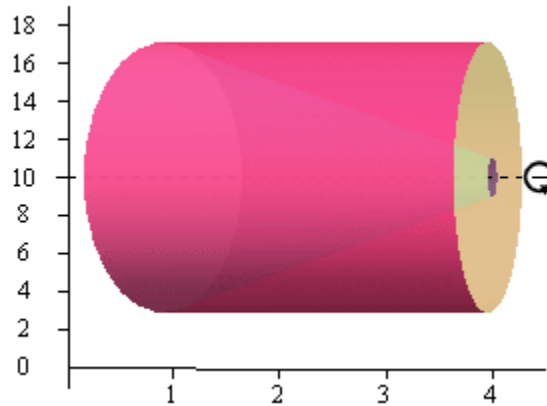Here is a sketch of the bounded region with the axis of rotation shown.

Hint : Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative cylinder.
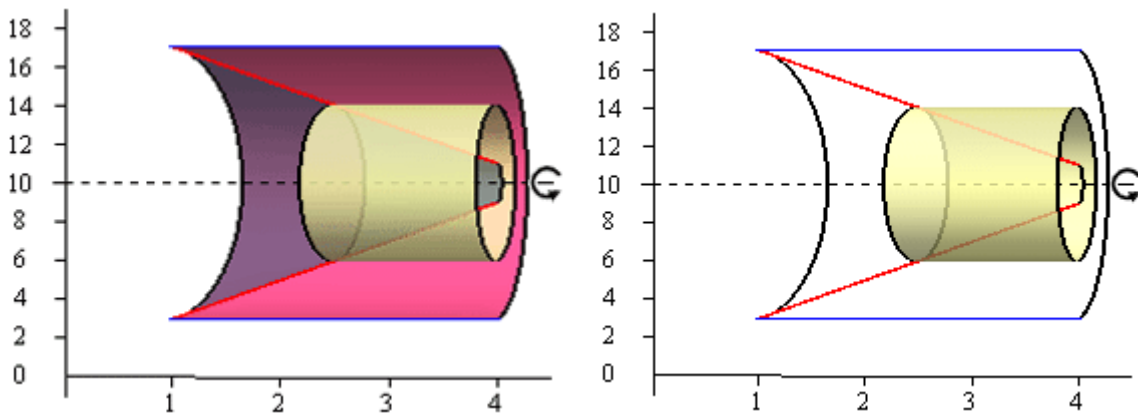
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative cylinder can be of great help when we go to write down the area formula. Also, getting the representative cylinder can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2
Here is a sketch of the solid of revolution.

Here are a couple of sketches of a representative cylinder. The image on the left shows a representative cylinder with the front half of the solid cut away and the image on the right shows a representative cylinder with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).
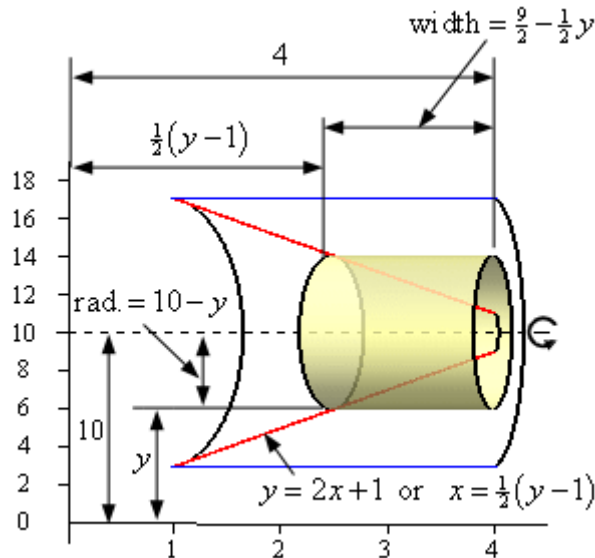

Hint : Determine a formula for the area of the cylinder.

Step 3

We now need to find a formula for the area of the cylinder. Because we are using cylinders that are centered on a horizontal axis (*i.e.* parallel to the *x*-axis) we know that the area formula will need to be in terms of *y*. Therefore we'll need to rewrite the equations of the curves in terms of *y*.

Here is another sketch of a representative cylinder with all of the various quantities we need put into it.

From the sketch we can see the cylinder is centered on the line $y = 10$ and the lower edge of the cylinder is at some *y*.

The radius of the cylinder is just the distance from the axis of rotation to the lower edge of the cylinder (*i.e.* $10 - y$).

The right edge of the cylinder is on the curve defining the right portion of the solid and is a distance of 4 from the *y*-axis. The left edge of the cylinder is on the curve defining the left portion of the solid and is a distance of $\frac{1}{2}(y-1)$ from the *y*-axis. The width then is the difference of these two.

So the radius and width of the cylinder are,

$$\text{Radius} = 10 - y \qquad\qquad \text{Width} = 4 - \tfrac{1}{2}(y-1) = \tfrac{9}{2} - \tfrac{1}{2}y$$

The area of the cylinder is then,

$$A(y) = 2\pi(\text{Radius})(\text{Height}) = 2\pi(10-y)\left(\tfrac{9}{2} - \tfrac{1}{2}y\right) = 2\pi\left(45 - \tfrac{19}{2}y + \tfrac{1}{2}y^2\right)$$

Step 4
The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" cylinder in the solid would occur at $y = 3$ and the "last" cylinder would occur at $y = 9$. Our limits are then : $3 \le y \le 9$.

The volume is then,

$$V = \int_3^9 2\pi \left(45 - \tfrac{19}{2}y + \tfrac{1}{2}y^2\right)dy = 2\pi\left(45y - \tfrac{19}{4}y^2 + \tfrac{1}{6}y^3\right)\Big|_3^9 = \boxed{90\pi}$$

6. Use the method of cylinders to determine the volume of the solid obtained by rotating the region bounded by $x = y^2 - 4$ and $x = 6 - 3y$ about the line $y = -8$.
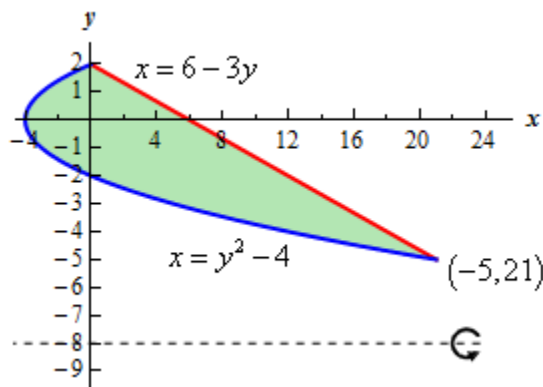
Hint : Start with sketching the bounded region.

Step 1
We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.

To get the intersection points shown above, which we'll need in a bit, all we need to do is set the two equations equal and solve.
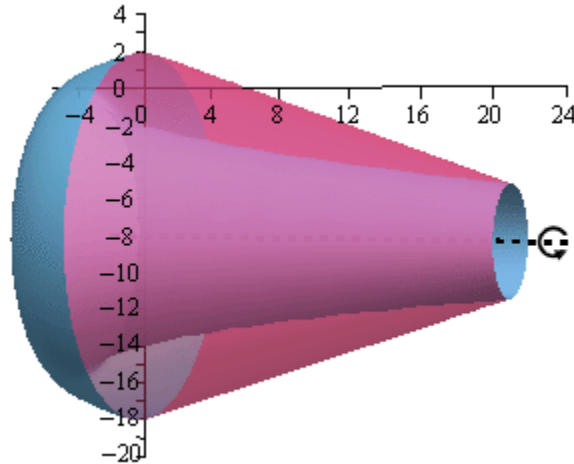
$$y^2 - 4 = 6 - 3y$$
$$y^2 + 3y - 10 = 0$$
$$(y+5)(y-2) = 0 \qquad \Rightarrow \qquad y = -5, \ y = 2 \qquad \Rightarrow \qquad (-5, 21) \ \& \ (2, 0)$$

Hint : Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative cylinder.
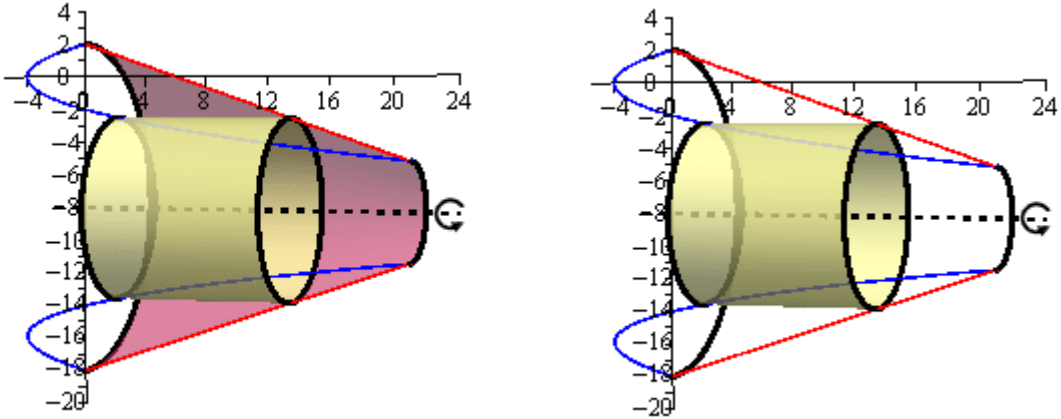
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative cylinder can be of great help when we go to write down the area formula. Also, getting the representative cylinder can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2
Here is a sketch of the solid of revolution.

Here are a couple of sketches of a representative cylinder. The image on the left shows a representative cylinder with the front half of the solid cut away and the image on the right shows a representative cylinder with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).
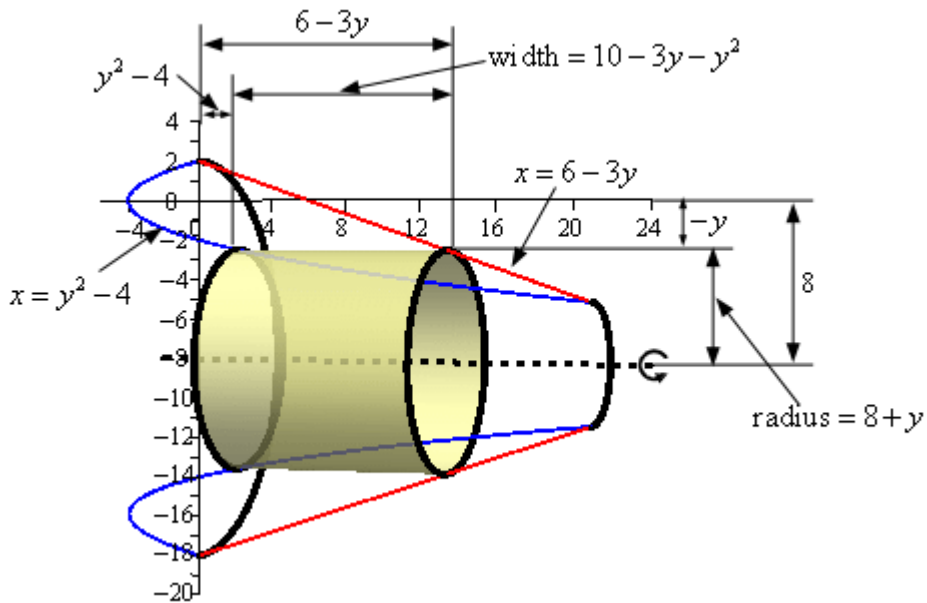
Hint : Determine a formula for the area of the cylinder.

Step 3
We now need to find a formula for the area of the cylinder. Because we are using cylinders that are centered on a horizontal axis (*i.e.* parallel to the *x*-axis) we know that the area formula will need to be in terms of *y*. Therefore the equations of the curves will need to be in terms of *y* (which in this case they already are).

Here is another sketch of a representative cylinder with all of the various quantities we need put into it.

From the sketch we can see the cylinder is centered on the line $y = -8$ and the upper edge of the cylinder is at some *y*.

The radius of the cylinder is a little tricky for this problem.

First, notice that the axis of rotation is a distance of 8 below the *x*-axis. Next, the upper edge of the cylinder is at some *y* however because *y* is negative at the point where we drew the cylinder that means that the distance of the upper edge below the *x*-axis is in fact $-y$. The minus in needed to turn this into a positive quantity that we need for distance. The radius for this cylinder is then the difference of these two distances or,

$$8-(-y)=8+y$$

Now, note that when the upper edge of the cylinder rises above the *x*-axis the distance of the upper edge above the *x*-axis will be just *y*. This time because *y* is positive we don't need the minus sign (and in fact don't want it because that would turn the distance into a negative quantity). The radius is then the distance of the axis of rotation from the *x*-axis (still a distance of 8) plus by the distance of the upper edge above the *x*-axis (which is *y*) or,

$$8+y$$

In either case we get the same radius.

The right edge of the cylinder is on the curve defining the right portion of the solid and is a distance of $6-3y$ from the *y*-axis. The left edge of the cylinder is on the curve defining the left portion of the solid and is a distance of y^2-4 from the *y*-axis. The width then is the difference of these two.

So the radius and width of the cylinder are,

$$\text{Radius} = 8+y \qquad \text{Width} = 6-3y-\left(y^2-4\right)=10-3y-y^2$$

The area of the cylinder is then,

$$A(y)=2\pi\left(\text{Radius}\right)\left(\text{Height}\right)=2\pi\left(8+y\right)\left(10-3y-y^2\right)=2\pi\left(80-14y-11y^2-y^3\right)$$

Step 4
The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" cylinder in the solid would occur at $y=-5$ and the "last" cylinder would occur at $y=2$. Our limits are then : $-5 \le y \le 2$.

The volume is then,

$$V = \int_{-5}^{2} 2\pi \left(80 - 14y - 11y^2 - y^3\right) dy = 2\pi \left(80y - 7y^2 - \tfrac{11}{3}y^3 - \tfrac{1}{4}y^4\right)\Big|_{-5}^{2} = \boxed{\tfrac{4459}{6}\pi}$$

---

7. Use the method of cylinders to determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 6x + 9$ and $y = -x^2 + 6x - 1$ about the line $x = 8$.
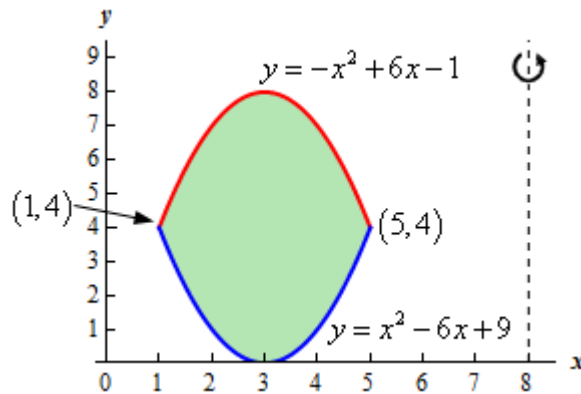
Hint : Start with sketching the bounded region.

Step 1
We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

Here is a sketch of the bounded region with the axis of rotation shown.

To get the intersection points shown above, which we'll need in a bit, all we need to do is set the two equations equal and solve.
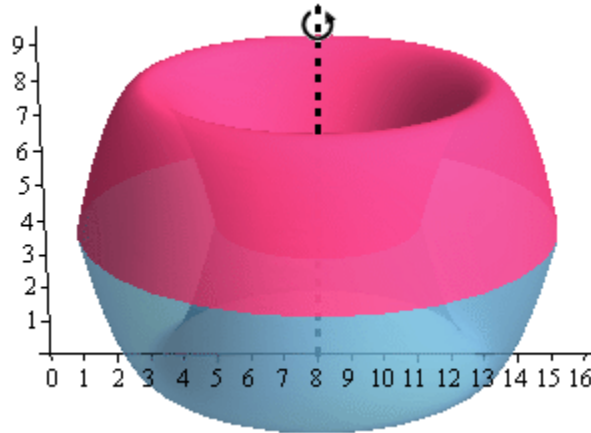
$$x^2 - 6x + 9 = -x^2 + 6x - 1$$
$$2x^2 - 12x + 10 = 0$$
$$2(x-1)(x-5) = 0 \quad \Rightarrow \quad x = 1, \ x = 5 \quad \Rightarrow \quad (1,4) \ \& \ (5,4)$$

Hint : Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative cylinder.
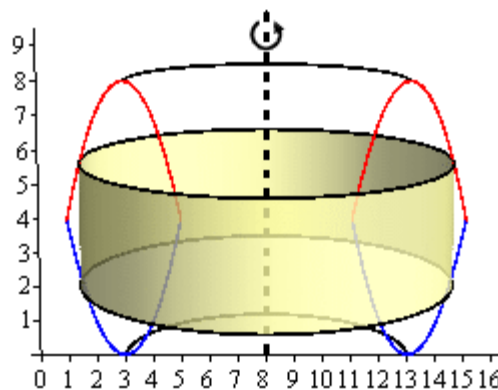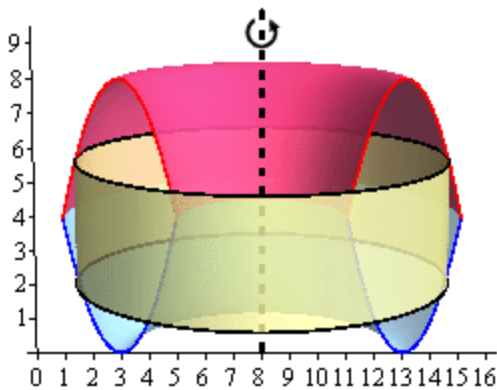
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative cylinder can be of great help when we go to write down the area formula. Also, getting the representative cylinder can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2
Here is a sketch of the solid of revolution.

Here are a couple of sketches of a representative cylinder. The image on the left shows a representative cylinder with the front half of the solid cut away and the image on the right shows a representative cylinder with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).
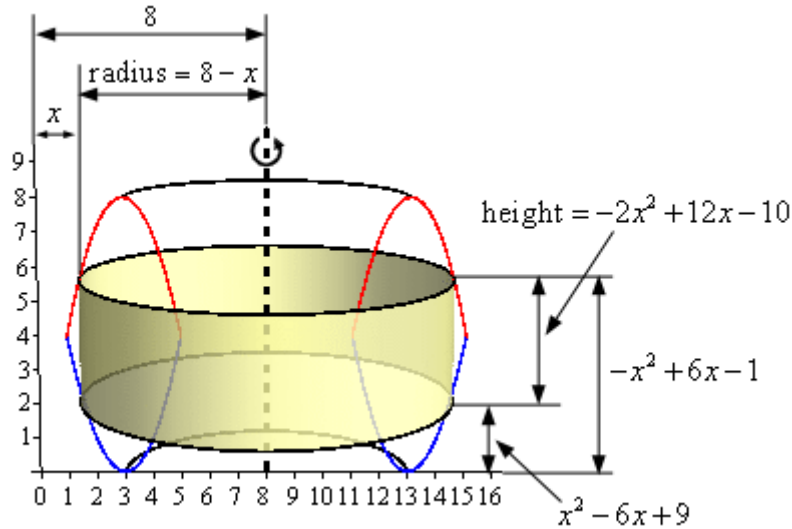

Hint : Determine a formula for the area of the cylinder.

Step 3
We now need to find a formula for the area of the cylinder. Because we are using cylinders that are centered on a vertical axis (*i.e.* parallel to the *y*-axis) we know that the area formula will need

to be in terms of x. Therefore the equations of the curves will need to be in terms of x (which in this case they already are).

Here is another sketch of a representative cylinder with all of the various quantities we need put into it.

Note that we put all the height "lines" on the mirrored curves and not the actual curves. This was done so we could put them in a place that didn't interfere with the y-axis.

From the sketch we can see the cylinder is centered on the line $x = 8$ and the left edge of the cylinder is at some x.

The radius of the cylinder is just the distance from the axis of rotation to the left edge of the cylinder (*i.e.* $8 - x$).

The upper edge of the cylinder is on the curve defining the upper portion of the solid and is a distance of $-x^2 + 6x - 1$ from the x-axis. The lower edge of the cylinder is on the curve defining the lower portion of the solid and is a distance of $x^2 - 6x + 9$ from the x-axis. The height then is the difference of these two.

So the radius and width of the cylinder are,

$$\text{Radius} = 8 - x \qquad \text{Width} = -x^2 + 6x - 1 - \left(x^2 - 6x + 9 \right) = -2x^2 + 12x - 10$$

The area of the cylinder is then,

$$A(x) = 2\pi(\text{Radius})(\text{Height})$$
$$= 2\pi(8-x)(-2x^2+12x-10) = 2\pi(-80+106x-28x^2+2x^3)$$

Step 4
The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" cylinder in the solid would occur at $x=1$ and the "last" cylinder would occur at $x=5$. Our limits are then : $1 \le x \le 5$.

The volume is then,

$$V = \int_1^5 2\pi(-80+106x-28x^2+2x^3)\,dy = 2\pi\left(-80x+53x^2-\tfrac{28}{3}x^3+\tfrac{1}{2}x^4\right)\Big|_1^5 = \boxed{\tfrac{640}{3}\pi}$$

---

8. Use the method of cylinders to determine the volume of the solid obtained by rotating the region bounded by $y = \dfrac{e^{\frac{1}{2}x}}{x+2}$, $y = 5-\tfrac{1}{4}x$, $x=-1$ and $x=6$ about the line $x=-2$.
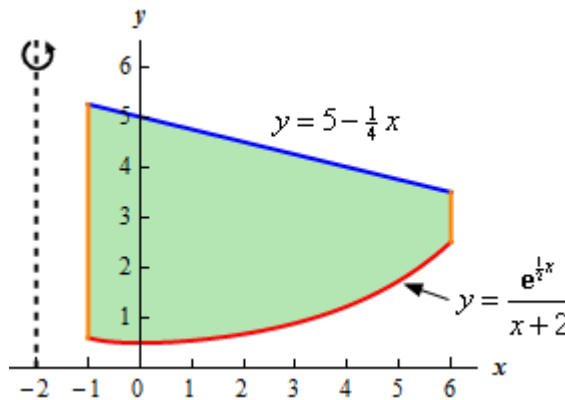
Hint : Start with sketching the bounded region.

Step 1
We need to start the problem somewhere so let's start "simple".

Knowing what the bounded region looks like will definitely help for most of these types of problems since we need to know how all the curves relate to each other when we go to set up the area formula and we'll need limits for the integral which the graph will often help with.

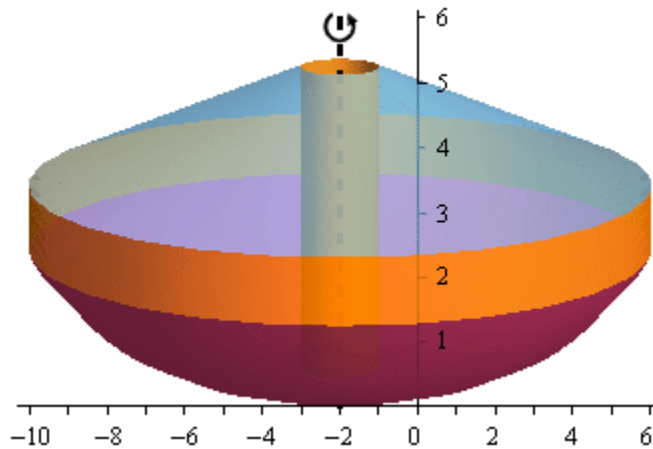Here is a sketch of the bounded region with the axis of rotation shown.

Hint : Give a good attempt at sketching what the solid of revolution looks like and sketch in a representative cylinder.
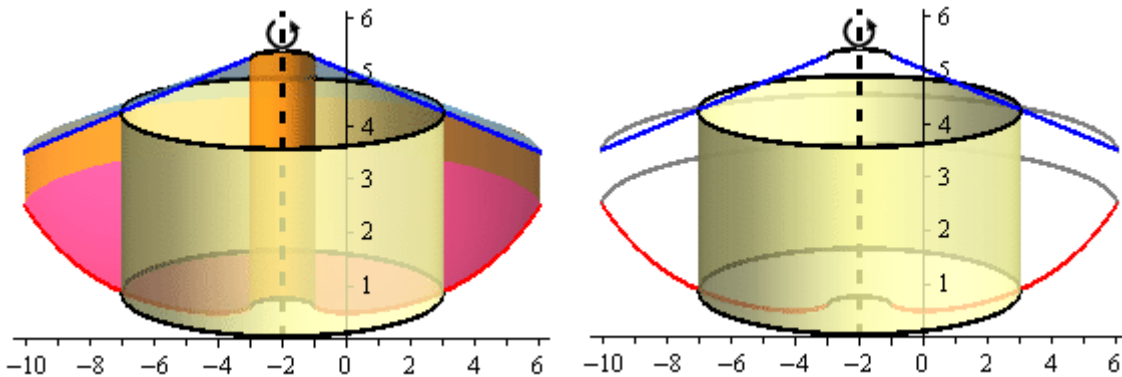
Note that this can be a difficult thing to do especially if you aren't a very visual person. However, having a representative cylinder can be of great help when we go to write down the area formula. Also, getting the representative cylinder can be difficult without a sketch of the solid of revolution. So, do the best you can at getting these sketches.

Step 2
Here is a sketch of the solid of revolution.

Here are a couple of sketches of a representative cylinder. The image on the left shows a representative cylinder with the front half of the solid cut away and the image on the right shows a representative cylinder with a "wire frame" of the back half of the solid (*i.e.* the curves representing the edges of the of the back half of the solid).
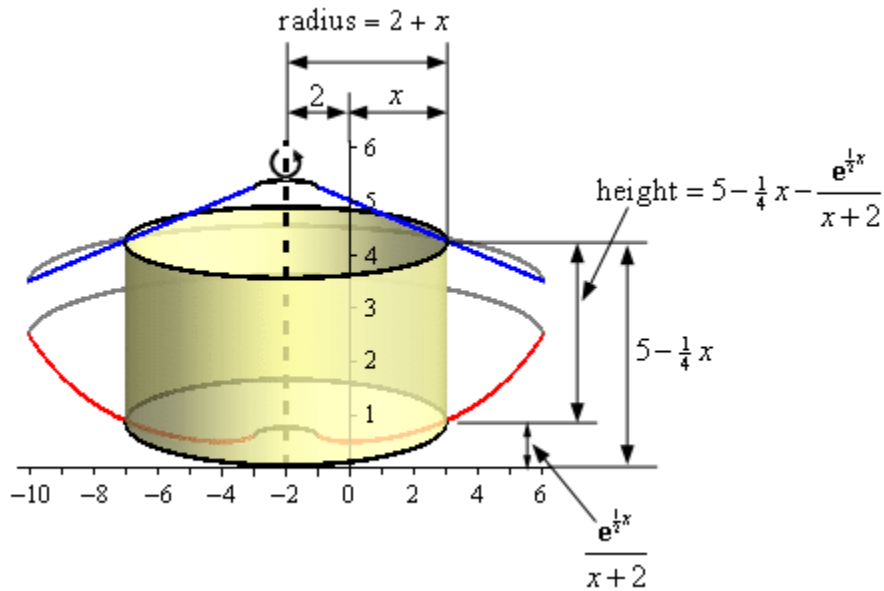

Hint : Determine a formula for the area of the cylinder.

Step 3

We now need to find a formula for the area of the cylinder. Because we are using cylinders that are centered on a vertical axis (*i.e.* parallel to the *y*-axis) we know that the area formula will need to be in terms of *x*. Therefore the equations of the curves will need to be in terms of *x* (which in this case they already are).

Here is another sketch of a representative cylinder with all of the various quantities we need put into it.

From the sketch we can see the cylinder is centered on the line $x = -2$ and the right edge of the cylinder is at some *x*.

We need to be a little careful with the radius here since the right edge of the cylinder can be on both the left and right side of the *y*-axis depending on where it cuts into the solid.

If the right edge of the cylinder cuts into the object to the right of the *y*-axis, as shown in the sketch above, then the radius is the distance of the axis of rotation to the *y*-axis (a distance of 2) plus the distance from the *y*-axis to the right edge of the cylinder (a distance of *x*). Therefore, in this case, the radius is $2 + x$.

On the other hand, if the right edge of the cylinder cuts into the solid to the left of the *y*-axis then the radius will be the distance from the axis of rotation to the *y*-axis (a distance of 2) minus the distance of the right edge of the cylinder to the *y*-axis. However, in this case, the value of the *x* that defines the right edge is a negative value and so the distance of the right edge of the cylinder to the *y*-axis must be $-x$. The minus sign in needed to turn this into a positive quantity.

Therefore the radius in this case is $2 - (-x) = 2 + x$, the same as in the first case.

The upper edge of the cylinder is on the curve defining the upper portion of the solid and is a distance of $5 - \frac{1}{4}x$ from the x-axis. The lower edge of the cylinder is on the curve defining the lower portion of the solid and is a distance of $\dfrac{\mathbf{e}^{\frac{1}{2}x}}{x+2}$ from the x-axis. The height then is the difference of these two.

So the radius and width of the cylinder are,

$$\text{Radius} = 2 + x \qquad\qquad \text{Width} = 5 - \tfrac{1}{4}x - \frac{\mathbf{e}^{\frac{1}{2}x}}{x+2}$$

The area of the cylinder is then,

$$A(x) = 2\pi(\text{Radius})(\text{Height}) = 2\pi(2+x)\left(5 - \tfrac{1}{4}x - \frac{\mathbf{e}^{\frac{1}{2}x}}{x+2}\right) = 2\pi\left(10 + \tfrac{9}{2}x - \tfrac{1}{4}x^2 - \mathbf{e}^{\frac{1}{2}x}\right)$$

Step 4
The final step is to then set up the integral for the volume and evaluate it.

From the graph from Step 1 we can see that the "first" cylinder in the solid would occur at $x = -1$ and the "last" cylinder would occur at $x = 6$. Our limits are then : $-1 \le x \le 6$.

The volume is then,

$$V = \int_{-1}^{6} 2\pi\left(10 + \tfrac{9}{2}x - \tfrac{1}{4}x^2 - \mathbf{e}^{\frac{1}{2}x}\right) dy$$

$$= 2\pi\left(10x + \tfrac{9}{4}x^2 - \tfrac{1}{12}x^3 - 2\mathbf{e}^{\frac{1}{2}x}\right)\Big|_{-1}^{6} = \boxed{2\pi\left(\tfrac{392}{3} + 2\mathbf{e}^{-\frac{1}{2}} - 2\mathbf{e}^3\right)}$$

More Volume Problems

1. Find the volume of a pyramid of height h whose base is an equilateral triangle of length L.

Hint : If possible, try to get a sketch of what the pyramid looks like. These can be difficult to sketch on occasion but if we can get a sketch it will help to set up the problem.

Step 1
Okay, let's start with a sketch of the pyramid. These can be difficult to sketch, but having the sketch will help greatly with the set up portion of the problem.
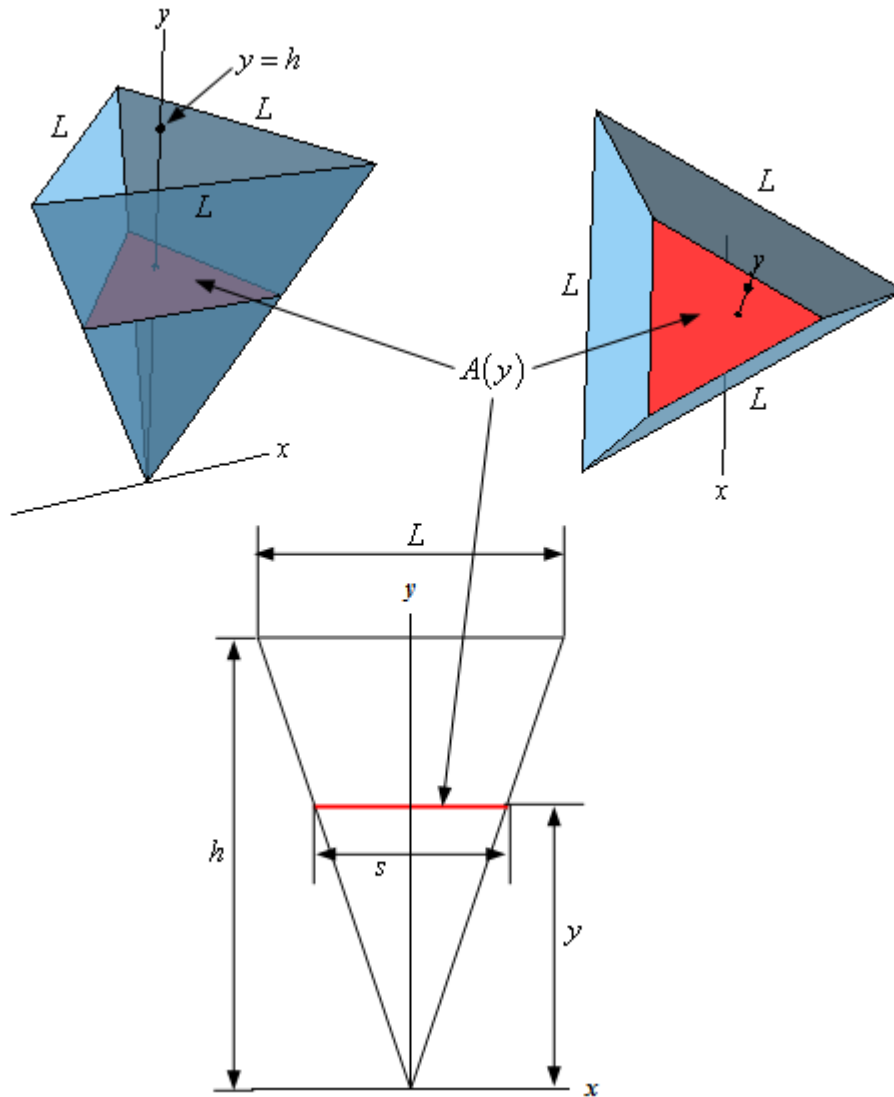
We've got several sketches here. In each sketch we've shown a representative cross-sectional area (shown in red). Because the cross-section can be placed at any point on the y-axis the area of the cross-section will be a function of y as indicated in the image.

The sketch in the upper right we see the pyramid from the "front" and the sketch in the upper left we see pyramid from the "top". Note that we set the point of the pyramid at the origin and drew the pyramid upwards. This was done to make the set up for the problem a little easier. Also we sketched the pyramid so that one of the sides of the pyramid was parallel to the x-axis. This was done only so we could draw in the bottom sketch (which we'll get to in a second) and have the images match up, so to speak.

The bottom sketch is a sketch of the side of the pyramid that is parallel to the *x*-axis. It also has all of the various quantities that we'll need shown. The representative cross-section here is indicated by the red line on the sketch.

Hint : Determine a formula for the cross-sectional area in terms of *y*.

Step 2
Let's start off with a sketch of what a typical cross-section looks like.
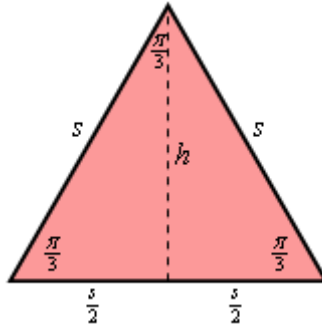
In this case we know that the cross-sections are equilateral triangles and so all of the interior angles are $\frac{\pi}{3}$ and we know that all the sides are the same length, let's say *s*. In the sketch above notice that since we have an equilateral triangle we know that the dashed line (representing the height of the triangle) will divide the base of the triangle into equal length portions, *i.e.* $\frac{s}{2}$. Also from basic right triangle trig (each "half" of the cross-section is a right triangle right?) we can see that we can write the height in terms of *s* as follows,

$$\tan\left(\tfrac{\pi}{3}\right) = \tfrac{h}{s/2} \qquad \Rightarrow \qquad h = \tfrac{s}{2}\tan\left(\tfrac{\pi}{3}\right) = \tfrac{\sqrt{3}}{2}s$$

Therefore, in terms of *s* the area of each cross-section is,

$$\text{Area} = \tfrac{1}{2}(s)\left(\tfrac{\sqrt{3}}{2}s\right) = \tfrac{\sqrt{3}}{4}s^2$$

Now, we know from the sketches in Step 1 that the cross-sectional area should be a function of *y*. So, if we could determine a relationship between *s* and *y* we'd have what we need. Let's revisit one of the sketches from Step 1.
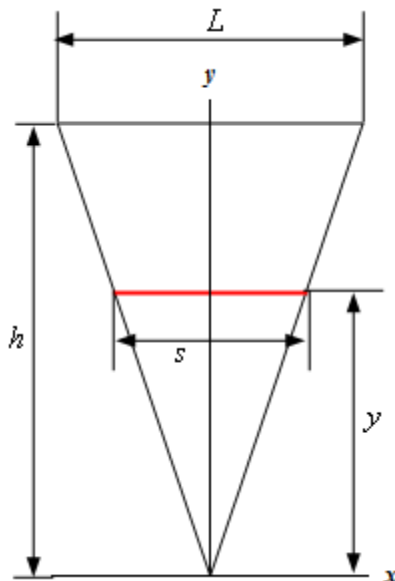
From this we can see that we have two similar triangles. The overall side (base L and height h) as well as the "lower" portion formed by the red line representing the cross-sectional area (base s and height y).

Because these two triangles are similar triangles we know the following ratios must be equal.

$$\frac{s}{y} = \frac{L}{h} \qquad \Rightarrow \qquad s = \frac{L}{h}y$$

We now have a relationship between s and y so plug this into the area formula from above to get the area of the cross-section in terms of y.

$$A(y) = \frac{\sqrt{3}}{4}\left(\frac{L}{h}y\right)^2 = \frac{\sqrt{3}L^2}{4h^2}y^2$$

Hint : All we need to do now is determine the volume itself.

Step 3
Finally we need the volume itself. We know that the volume is found by evaluating the following integral.

$$V = \int_c^d A(y)\,dy$$

We already have a formula for $A(y)$ from Step 2 and from the sketches in Step 1 we can see that the "first" cross-section will occur at $y = 0$ and that the "last" cross-section will occur at $y = h$ and so these are the limits for the integral.

The volume is then,

$$V = \int_0^h \frac{\sqrt{3}L^2}{4h^2} y^2 \, dy = \frac{\sqrt{3}L^2}{4h^2} \int_0^h y^2 \, dy = \frac{\sqrt{3}L^2}{4h^2} \left(\tfrac{1}{3} y^3 \right)\Big|_0^h = \boxed{\frac{\sqrt{3}\, L^2 h}{12}}$$

Do not get excited about the h and L in the integral and area formula. These are just constants. The only letter that is actually changing is y. Because the h and L are constants we can factor them out of the integral as we did with the actual numbers.

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2. Find the volume of the solid whose base is a disk of radius r and whose cross-sections are squares. See figure below to see a sketch of the cross-sections.
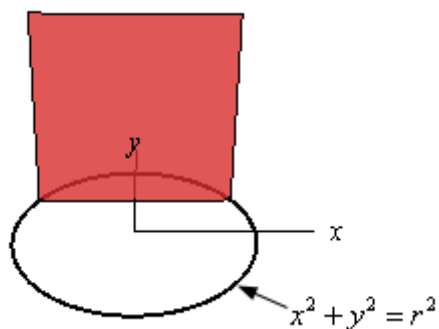
Hint : While it's not strictly needed for this problem a sketch of the solid might be interesting to see just what the solid looks like.

Step 1
Here are a couple of sketches of the solid from three different angles. For reference the positive x-axis and positive y-axis are shown.

Because the cross-section is perpendicular to the y-axis as we move the cross-section along the y-axis we'll change its area and so the cross-sectional area will be a function of y, *i.e.* $A(y)$.

Hint : Determine a formula for the cross-sectional area in terms of y.

Step 2
While the sketches above are nice to get a feel for what the solid looks like, what we really need is just a sketch of the cross-section. So, here's a couple of sketches of the cross-sectional area.
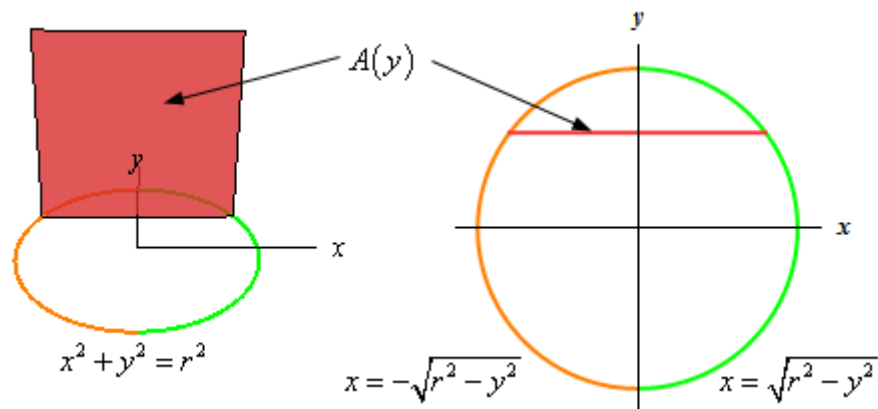
The sketch on the left is really just the graph given in the problem statement with the only difference that we colored the right/left sides so it will match with the sketch on the right. The sketch on the right looks at the cross-section from directly above and is shown by the red line.

Let's get a quick sketch of just the cross-section and let's call the length of the side of each square *s*.
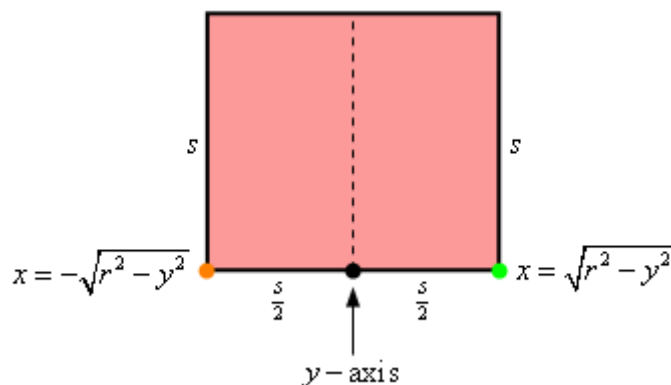
Now, along the bottom we've denoted the *y*-axis location in the cross-section with a black dot and the orange and green dots represent where the left and right portions of the circle are at. We can also see that, assuming the cross-section is placed at some *y*, the green dot must be a distance of $\sqrt{r^2 - y^2}$ from the *y*-axis. Likewise, the orange dot must also be a distance of $\sqrt{r^2 - y^2}$ from the *y*-axis (recall we want the distance to be positive here and so we drop the minus sign from the function to get a positive distance).

Now, we know that the area of the square is simply s^2 and from the discussion above we see that,

$$\tfrac{s}{2} = \sqrt{r^2 - y^2} \qquad \Rightarrow \qquad s = 2\sqrt{r^2 - y^2}$$

So, a formula for the area of the cross-section in terms of *y* is,

$$A(y) = s^2 = \left(2\sqrt{r^2 - y^2}\right)^2 = 4\left(r^2 - y^2\right)$$

Hint : All we need to do now is determine the volume itself.

Step 3
Finally we need the volume itself. We know that the volume is found by evaluating the following integral.
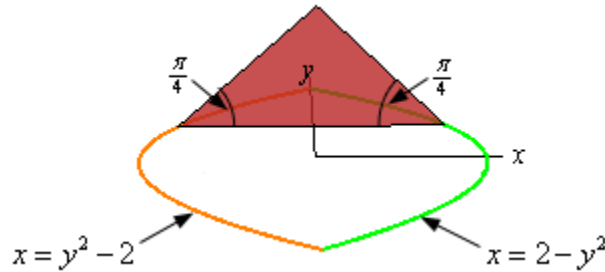
$$V = \int_c^d A(y)\, dy$$

We already have a formula for $A(y)$ from Step 2 and from the sketches in Step 1 we can see that the "first" cross-section will occur at $y = -r$ and that the "last" cross-section will occur at $y = r$ and so these are the limits for the integral.

The volume is then,

$$V = \int_{-r}^{r} 4(r^2 - y^2) dy = 4\left(yr^2 - \tfrac{1}{3}y^3\right)\Big|_{-r}^{r} = \boxed{\tfrac{16}{3}r^3}$$

Do not get excited about the r integral and area formula. It is just a constant. The only letter that is actually changing is y.

---

3. Find the volume of the solid whose base is the region bounded by $x = 2 - y^2$ and $x = y^2 - 2$ and whose cross-sections are isosceles triangles with the base perpendicular to the y-axis and the angle between the base and the two sides of equal length is $\frac{\pi}{4}$. See figure below to see a sketch of the cross-sections.

Hint : While it's not strictly needed for this problem a sketch of the solid might be interesting to see just what the solid looks like.

Step 1
Here are a couple of sketches of the solid from three different angles. For reference the positive x-axis and positive y-axis are shown.

$A(y)$

Because the cross-section is perpendicular to the *y*-axis as we move the cross-section along the *y*-axis we'll change its area and so the cross-sectional area will be a function of *y*, *i.e.* $A(y)$.

Hint : Determine a formula for the cross-sectional area in terms of *y*.

Step 2
While the sketches above are nice to get a feel for what the solid looks like, what we really need is just a sketch of the cross-section. So, here's a couple of sketches of the cross-sectional area.

The sketch on the top is really just the graph given in the problem statement that is included for a reference with the sketch on the bottom. The sketch on the bottom looks at the cross-section from directly above and is shown by the red line.

Let's get a quick sketch of just the cross-section and let's call the length of the base of triangle b and the height of the triangle h.
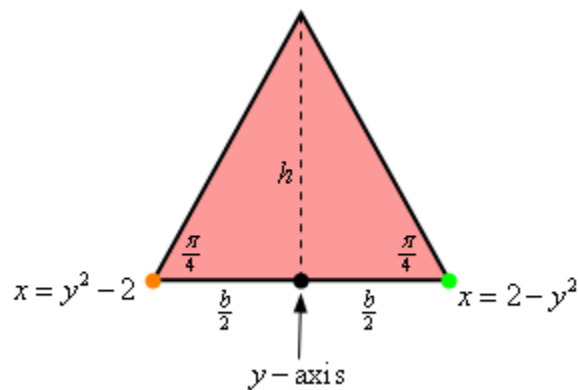
Now, along the bottom we've denoted the y-axis location in the cross-section with a black dot and the orange and green dots represent the left and right curves that define the left and right sides of the bottom of the solid. We can also see that, assuming the cross-section is placed at some y, the

green dot must be a distance of $2-y^2$ from the y-axis. Likewise, the orange dot must also be a distance of $-\left(y^2-2\right)=2-y^2$ from the y-axis (recall we want the distance to be positive here and so we add the minus sign to the function to get a positive distance).

Now, we can see that the base of the triangle is given by,

$$\tfrac{b}{2}=2-y^2 \qquad\qquad \Rightarrow \qquad\qquad b=2\left(2-y^2\right)$$

Likewise, the height can be found from basic right triangle trig.

$$\tan\left(\tfrac{\pi}{4}\right)=\dfrac{h}{b/2} \qquad\qquad \Rightarrow \qquad\qquad h=\tfrac{b}{2}\tan\left(\tfrac{\pi}{4}\right)=2-y^2$$

So, a formula for the area of the cross-section in terms of y is then,

$$A(y)=\tfrac{1}{2}bh=\left(2-y^2\right)^2=4-4y^2+y^4$$

Hint : All we need to do now is determine the volume itself.

Step 3
Finally we need the volume itself. We know that the volume is found by evaluating the following integral.

$$V=\int_c^d A(y)\,dy$$

By setting $x=0$ into either of the equations defining the left and right sides of the base of the solid (since they intersect at the y-axis) we can see that the "first" cross-section will occur at $y=-\sqrt{2}$ and the "last" cross-section will occur at $y=\sqrt{2}$ and so these are the limits for the integral.

The volume is then,

$$V=\int_{-\sqrt{2}}^{\sqrt{2}}4-4y^2+y^4\,dy=\left(4y-\tfrac{4}{3}y^3+\tfrac{1}{5}y^5\right)\Big|_{-\sqrt{2}}^{\sqrt{2}}=\boxed{\tfrac{64\sqrt{2}}{15}}$$

---

4. Find the volume of a wedge cut out of a "cylinder" whose base is the region bounded by $y=\sqrt{4-x}$, $x=-4$ and the x-axis. The angle between the top and bottom of the wedge is $\tfrac{\pi}{3}$.
See the figure below for a sketch of the "cylinder" and the wedge (the positive x-axis and positive y-axis are shown in the sketch – they are just in a different orientation).

"Cylinder"

Wedge

Step 1

While not strictly needed let's redo the sketch of the "cylinder" and wedge from the problem statement only this time let's also sketch in what the cross-section will look like.
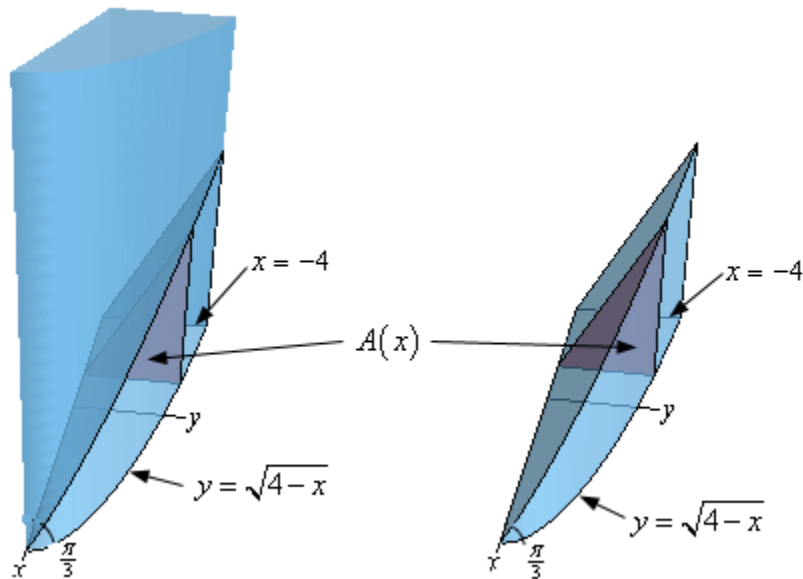
Because the cross-section is perpendicular to the *x*-axis as we move the cross-section along the *x*-axis we'll change its area and so the cross-sectional area will be a function of *x*, *i.e.* $A(x)$. Also note that as shown in the sketches the cross-section will be a right triangle.
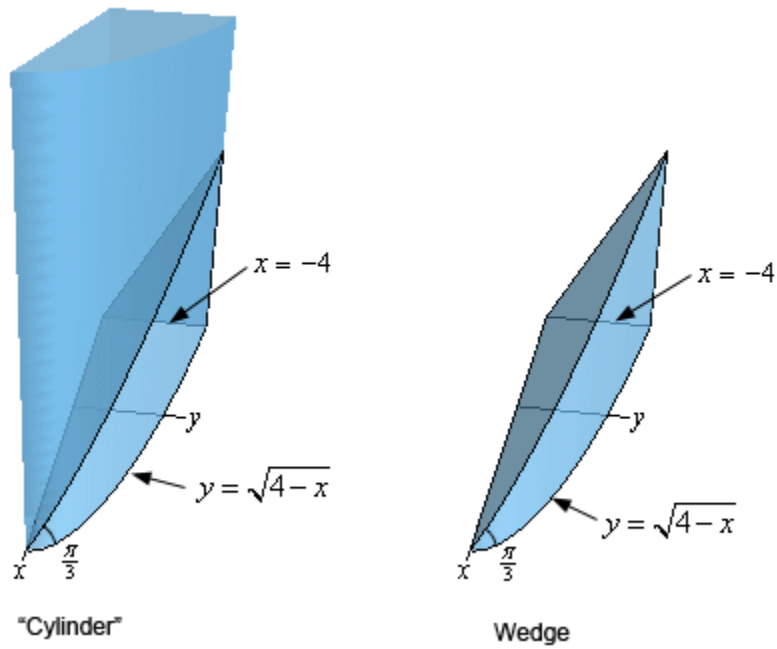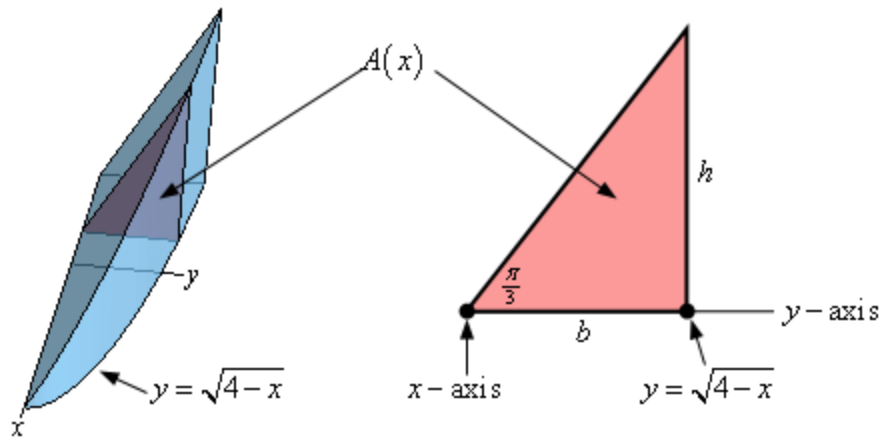
Hint : Determine a formula for the cross-sectional area in terms of *x*.

Step 2
While the sketches above are nice to get a feel for what the solid and cross-sections look like, what we really need is just a sketch of just the cross-section. So, here are a couple of sketches of the cross-sectional area.

The sketch on the left is just pretty much the sketch we've seen before and is included to give us a reference point for the actual cross-section that is shown on the right.

As noted in the sketch on the right we'll call the base of the triangle *b* and the height of the triangle *h*. Also, the dot on the left side of the base represents where the *x*-axis is on the cross-section and the dot on the right side of the base represents the curve that defines the edge of the solid (and hence the wedge).

From this sketch it should then be pretty clear that the length of the base is simply the distance from the *x*-axis to the curve or,

$$b = \sqrt{4 - x}$$

Likewise, the height can be found from basic right triangle trig.

$$\tan\left(\tfrac{\pi}{3}\right) = \tfrac{h}{b} \qquad\Rightarrow\qquad h = b\tan\left(\tfrac{\pi}{3}\right) = \sqrt{3}\sqrt{4 - x}$$

So, a formula for the area of the cross-section in terms of *x* is then,

$$A(y) = \tfrac{1}{2}bh = \tfrac{\sqrt{3}}{2}\left(\sqrt{4-x}\right)^2 = \tfrac{\sqrt{3}}{2}(4-x)$$

Hint : All we need to do now is determine the volume itself.

Step 3
Finally we need the volume itself. We know that the volume is found by evaluating the following integral.

$$V = \int_a^b A(x)\,dx$$

From the sketches in the problem statement or from Step 1 we can see that the "first" cross-section will occur at $x = -4$ (the back end of the "cylinder") and the "last" cross-section will occur at $x = 4$ (the front end of the "cylinder" where the curve intersects with the x-axis. These are then the limits for the integral.

The volume is then,

$$V = \int_{-4}^{4} \tfrac{\sqrt{3}}{2}(4-x)\,dx = \tfrac{\sqrt{3}}{2}\left(4x - \tfrac{1}{2}x^2\right)\Big|_{-4}^{4} = \boxed{16\sqrt{3}}$$

Work

1. A force of $F(x) = x^2 - \cos(3x) + 2$, x is in meters, acts on an object. What is the work required to move the object from $x = 3$ to $x = 7$?

Solution
There really isn't all that much to this problem. We are given the force function and limits for the integral ($x = 3$ and $x = 7$) and so all we need to do is write down the integral for the work and evaluate it.

$$W = \int_3^7 x^2 - \cos(3x) + 2\,dx$$

$$= \left(\tfrac{1}{3}x^3 - \tfrac{1}{3}\sin(3x) + 2x\right)\Big|_3^7 = \boxed{\tfrac{1}{3}\left(340 + \sin(9) - \sin(21)\right) = 113.1918}$$

---

2. A spring has a natural length of 18 inches and a force of 20 lbs is required to stretch and hold the spring to a length of 24 inches. What is the work required to stretch the spring from a length of 21 inches to a length of 26 inches?

Hint : What is the spring constant, k and the force function?

Step 1
Let's start off by finding the spring constant. We are told that a force of 20 lbs is needed to stretch the spring 24 in – 18 in = 6 in = 0.5 ft from its natural length. Then using Hooke's Law we have,

$$20 = k(0.5) \qquad \Rightarrow \qquad k = 40$$

Don't forget that we want the displacement in feet. Also don't forget that the displacement needs to be the displacement from the natural length of the spring.

Again, using Hooke's Law we can see that the force function is,

$$F(x) = 40x$$

Step 2
For the limits of the integral we can see that we start with the spring at a length of 21 in – 18 in = 3 in or $\frac{1}{4}$ feet and we end with a length of 26 in – 18 in = 8 in or $\frac{2}{3}$ feet. These are then the limits of the integral (recall that we need the relative distance from the natural length for the limits).

The work is then,

$$W = \int_{\frac{1}{4}}^{\frac{2}{3}} 40x\,dx = 20x^2 \Big|_{\frac{1}{4}}^{\frac{2}{3}} = \boxed{\frac{275}{36} = 7.6389\,\text{ft-lbs}}$$

---

3. A cable that weighs ½ kg/meter is lifting a load of 150 kg that is initially at the bottom of a 50 meter shaft. How much work is required to lift the load ¼ of the way up the shaft?

Hint : What is the total mass of the chain and load at any point in the shaft? How does that relate to the force required to hold the chain and load at any point in the shaft?

Step 1
Let's start off with the convention that $x = 0$ defines the bottom of the shaft and $x = 50$ defines the top of the shaft. Therefore, x represents the distance that the load has been lifted. After lifting the load by x meters there will be $50 - x$ meters of the chain left in the shaft that needs to be lifted along with the load.

Therefore, after lifting the load x meters, the total mass of the chain left in the shaft as well as the load is,

$$\tfrac{1}{2}(50-x)+150\,\text{kg} = 175 - \tfrac{1}{2}x\,\text{kg}$$

We know that the force required to hold the chain and load at any point is just the total weight of the chain and load at that point. We also know that (because we are in the metric system) the weight of a given mass (in kg) is just then,

$$\text{Weight} = \text{mass} \times 9.8$$

where 9.8 is the gravitational acceleration.

The force required to hold the chain and load a distance of x meters above the bottom is then,

$$F(x) = (9.8)(175 - \tfrac{1}{2}x) = 1715 - 4.9x$$

Step 2
For the limits of the integral we can see that we start with the chain and load at the bottom of the shaft (*i.e.* at $x = 0$) and stop ¼ of the way up the shaft (*i.e.* at $x = 12.5$). These values are then the limits for the integral.

The work is then,

$$W = \int_0^{12.5} 1715 - 4.9x \, dx = \left(1715x - 2.45x^2\right)\Big|_0^{12.5} = \boxed{21{,}054.6875 J}$$

---

4. A tank of water is 15 feet long and has a cross section in the shape of an equilateral triangle with sides 2 feet long (point of the triangle points directly down). The tank is filled with water to a depth of 9 inches. Determine the amount of work needed to pump all of the water to the top of the tank. Assume that the density of water is 62 lb/ft³.

Hint : Get the basic problem set up. Determine all the known information and what you will need in order to work the problem. A sketch of at least the cross-section of the tank would probably be useful as well.

Use the last example from this section as a general guide for this problem if you are having trouble. This problem will work in pretty much the same manner, although there will be some differences due to the obvious change in tank shape as well as the fact that we are using not using the Metric system for this problem.

Step 1

Okay, let's start off and define $x = 0$ to be the bottom point of the tank and the height of the water in the tank to be $x = \frac{9}{12} = \frac{3}{4}$ feet (because all the other quantities are in feet we converted this into feet as well). This means that we will be working in the interval $\left[0, \frac{3}{4}\right]$ for this problem.

We'll next divide the interval $\left[0, \frac{3}{4}\right]$ into n subintervals each of width Δx and we'll let x_i^* be any point in the ith subinterval where $n = 1, 2, \ldots, n$. For each subinterval we can approximate the water in the tank corresponding to that subinterval as a box with length 15 ft, width s_i^* and height Δx.

Here is a quick sketch of the tank. The red strip represents the box we are using to approximate the water in the tank in the ith subinterval.
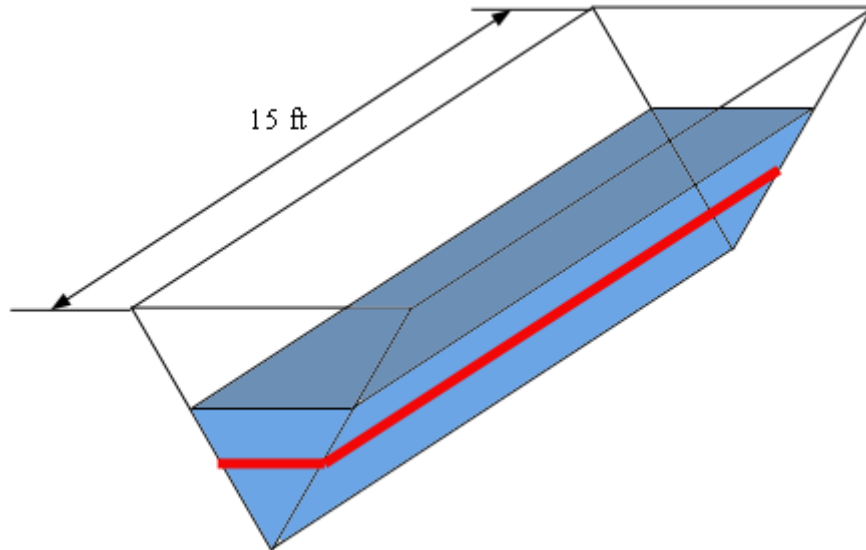

The sketch of the tank is nice and while it does help us to visualize the tank what we really need is a sketch of the tank from directly in front (*i.e.* a typical vertical equilateral triangular cross-section for the tank). Here is that sketch.
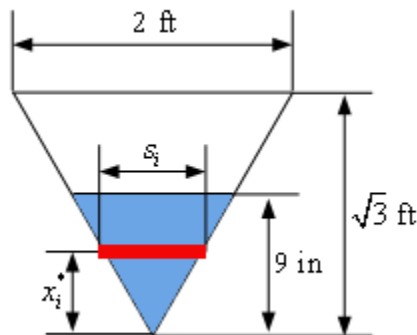
The red strip again represents the box we are using to approximate the water in the ith subinterval. As noted in the problem statement the cross-section is an equilateral triangle and with sides of length 2 feet.

We included the height in the above sketch and this is easy to get using some basic right triangle trig. Here is yet another sketch of the cross-section.
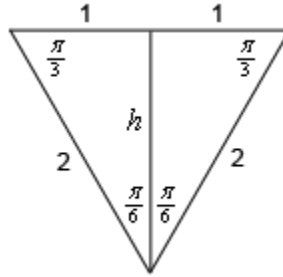
Because the triangle is an equilateral triangle we know that each of the interior angles of the triangle must be $\frac{\pi}{3}$ and we're told the length of each side is 2. The height of the triangle is the line that bisects the triangle as shown. Each half of the triangle is then an identical right triangle and using any of the trig functions we can quickly determine the height of the triangle. We'll use cosine here.

$$\cos\left(\tfrac{\pi}{6}\right) = \tfrac{h}{2} \qquad \Rightarrow \qquad h = 2\cos\left(\tfrac{\pi}{6}\right) = \sqrt{3}$$

Hint : What is the volume of the box of water we are using to approximate the volume of water in the ith subinterval? Give the volume in terms of x_i^*.

Step 2
We'll next need the volume of the box of water we using to approximate the volume of water in the ith subinterval (as represented by the red strip in the first two pictures from Step 1).

Our approximate volume is the volume of a box and so we know that the volume for the ith subinterval would be,

$$V_i = \left(length\right)\left(width\right)\left(height\right) = (15)\left(s_i\right)\left(\Delta x\right) = 15\, s_i\, \Delta x$$

We will eventually need the volume to be in terms of x_i^* and luckily enough this is easy enough to do.

From the cross-section sketch with the red strip in Step 1 we see that we have two similar triangles (well actually we have three but we only need two of them). The two that we need are the triangle with width 2 and height $\sqrt{3}$ and the triangle whose width is s_i (*i.e.* the triangle whose top is the red strip) and whose height is x_i^*. Since these two triangles are similar we now the following two ratios must be equal.

$$\frac{s_i}{x_i^*} = \frac{2}{\sqrt{3}} \qquad \Rightarrow \qquad s_i = \frac{2}{\sqrt{3}} x_i^*$$

Plugging this into the volume formula above and we get,

$$V_i = \tfrac{30}{\sqrt{3}} x_i^* \, \Delta x$$

Hint : What is the approximate weight of the water in the ith subinterval? Or in other words what is the approximate force needed to overcome the force of gravity acting on this volume of water?

Note that because we are working with the British system here the force in this case is just $F_i = \text{density} \times V_i$.

Step 3
We next need to know how much force will be required to overcome the force of gravity that is acting on the water in the ith subinterval. This will be approximately the forced needed to overcome the force of gravity acting on the volume of water we found in Step 2. Because we are working with the British system here the force is,

$$F_i = \text{density} \times V_i \approx \left(62\right)\left(\tfrac{30}{\sqrt{3}} x_i^* \, \Delta x\right) = \tfrac{1860}{\sqrt{3}} x_i^* \, \Delta x$$

Hint : Approximately how much work is needed to raise the water in the ith subinterval to the top of the tank?

Step 4
We will need the amount of work required to raise the volume of water in the ith subinterval to the top of the tank, *i.e.* raise it a distance of $\sqrt{3} - x_i^*$. This is approximately,

$$W_i \approx F_i\left(\sqrt{3} - x_i^*\right) = \tfrac{1860}{\sqrt{3}} x_i^* \left(\sqrt{3} - x_i^*\right)\Delta x$$

Hint : Finally compute the total amount of work needed to pump all the to the top of the tank.

Step 5

The total amount of work to raise all the water to the top of the tank is the approximately the sum of all the W_i for $i = 1, 2, \ldots n$ or,

$$W \approx \sum_{i=1}^{n} \tfrac{1860}{\sqrt{3}} x_i^* \left(\sqrt{3} - x_i^* \right) \Delta x$$

The exact work required is then found by letting $n \to \infty$ or,

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} \tfrac{1860}{\sqrt{3}} x_i^* \left(\sqrt{3} - x_i^* \right) \Delta x$$

This however is just the definition of the following definite integral,

$$W = \int_0^{\frac{3}{4}} \tfrac{1860}{\sqrt{3}} x \left(\sqrt{3} - x \right) dx$$

The work required to pump all the water to the top of the tank is then,

$$W = \int_0^{\frac{3}{4}} \tfrac{1860}{\sqrt{3}} x \left(\sqrt{3} - x \right) dx = \tfrac{1860}{\sqrt{3}} \left(\tfrac{\sqrt{3}}{2} x^2 - \tfrac{1}{3} x^3 \right) \Big|_0^{\frac{3}{4}} = \boxed{372.112 \text{ ft-lbs}}$$