Answer Key

Calculus 120

Worksheet – C1: Average and Instantaneous Rates of Change

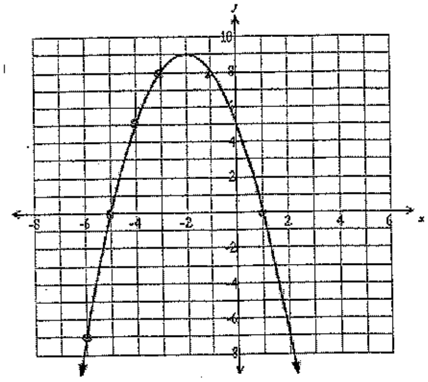
1 **Q** Given the graph below, determine the average rate of change over the following intervals.

a) $x = -4$ to $x = 1$

b) $x = -5$ to $x = -1$

c) $x = -6$ to $x = -3$

Answers: a) $m = -1$ b) $m = 2$ c) $m = 5$



2 **Q** Find the average rate of change of each function over the indicated interval.

a) $y = 4x - 3x^2, \ [2,3]$

b) $y = \sqrt{x}, \ [0,2]$

c) $y = \frac{2x+1}{x+2}, \ [1,3]$

Answers: a) $(2,-4)(3,-15)\ m = -11$

b) $(0,0)\ (2,\sqrt{2})\ m = \frac{\sqrt{2}}{2}$

c) $(1,1)\left(3,\frac{7}{5}\right)\ m = \frac{1}{5}$

3 **Q** An object dropped from rest from the top of a cliff falls $h(t) = -4.9t^2 + 50$ metres in the first t seconds.

a) Find the average speed during the first 3 seconds of fall. *(Answer:* $-14.7\ m/s$)

b) Find the speed of the object 3 seconds after it has been dropped. *(Answer:* $-29.4\ m/s$)

4 **Q** For each function at the indicated x-value, find

- the slope of the curve;
- the equation of the tangent line;
- the equation of the normal line.

a) $y = 4x - x^2, \ x = 1$ *Answer:* a) $m_{tan} = 2$ tan line $y = 2x + 1$, norm line $y = -\frac{1}{2}x + \frac{7}{2}$

b) $y = x - x^3, \ x = -1$ b) $m_{tan} = -2$ tan line $y = -2x - 2$, norm line $y = \frac{1}{2}x + \frac{1}{2}$

c) $y = \frac{2}{x+1}, \ x = -2$ c) $m_{tan} = -2$ tan line $y = -2x - 6$, norm line $y = \frac{1}{2}x - 1$

5 **Q** Find the instantaneous rate of change of the position function $y = t^2 - 8t$ in metres, at $t = 2$ seconds. *Answer:* $m = -4\ m/s$

6 **Q** What is the rate of change of the area of a square with respect to the length of a side when the length is $5\ cm$? *Answer:* $m = 10\ cm^2/cm$

7 **Q** Explain why you cannot find the equation of a tangent line to the curve $y = \frac{x+1}{x-2}$ at $x = 2$.

Answer: $x - 2 \neq 0$, $\therefore x \neq 2$ There is a vertical asymptote at $x = 2$

Q1 a) $x = -4$ to $x = 1$

$(-4, 5)$, $(1, 0)$

$$m_{sec} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{sec} = \frac{0 - 5}{1 - (-4)}$$

$$m_{sec} = \frac{-5}{5}$$

$$m_{sec} = -1$$

b) $x = -5$ to $x = -1$

$(-5, 0)$, $(-1, 8)$

$$m_{sec} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{sec} = \frac{8 - 0}{-1 - (-5)}$$

$$m_{sec} = \frac{8}{4}$$

$$m_{sec} = 2$$

c) $x = -6$ to $x = -3$

$(-6, -7)$, $(-3, 8)$

$$m_{sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - (-7)}{-3 - (-6)} = \frac{15}{3} = 5$$

Q2 a) $y = 4x - 3x^2$ $[2, 3]$

$f(2) = 4(2) - 3(2)^2 = 8 - 3(4) = 8 - 12 = -4$ $(2, -4)$

$f(3) = 4(3) - 3(3)^2 = 12 - 3(9) = 12 - 27 = -15$ $(3, -15)$

$m_{sec} = \frac{y_2 - y_1}{x_2 - x_1}$

$m_{sec} = \frac{-15 - (-4)}{3 - 2}$

$m_{sec} = \frac{-15 + 4}{1} = -11$

b) $y = \sqrt{x}$ $[0, 2]$

$f(0) = \sqrt{0} = 0$ $(0, 0)$

$f(2) = \sqrt{2}$ $(2, \sqrt{2})$

$m_{sec} = \frac{y_2 - y_1}{x_2 - x_1}$

$m_{sec} = \frac{\sqrt{2} - 0}{2 - 0} = \frac{\sqrt{2}}{2}$

c) $y = \frac{2x + 1}{x + 2}$ $[1, 3]$

$f(1) = \frac{2(1) + 1}{1 + 2} = \frac{3}{3} = 1$ $(1, 1)$

$f(3) = \frac{2(3) + 1}{3 + 2} = \frac{7}{5}$ $(3, \frac{7}{5})$

$m_{sec} = \frac{y_2 - y_1}{x_2 - x_1}$

$m_{sec} = \frac{\frac{7}{5} - 1}{3 - 1}$

$m_{sec} = \frac{\frac{7 - 5}{5}}{2} = \frac{\frac{2}{5}}{\frac{2}{1}}$

$m_{sec} = \frac{2}{5} \cdot \frac{1}{2}$

$m_{sec} = \frac{1}{5}$

Q3 $h(t) = -4.9t^2 + 50$

a) average speed during the first 3 seconds of fall.

$[0, 3]$

$h(0) = -4.9(0)^2 + 50 = 50$

$h(3) = -4.9(3)^2 + 50 = -44.1 + 50 = 5.9$

$(0, 50)$, $(3, 5.9)$

$V_{ave} = \frac{h_2 - h_1}{t_2 - t_1} = \frac{5.9 - 50}{3 - 0} = -14.7$ m/s

b) speed of the object 3 seconds after it has been dropped.

$t = 3$

$v(a) = \lim_{t \to a} \frac{h(t) - h(a)}{t - a}$

$V(3) = \lim_{t \to 3} \frac{h(t) - h(3)}{t - 3}$

$h(3) = -4.9(3)^2 + 50$

$h(3) = 5.9$

$V(3) = \lim_{t \to 3} \frac{-4.9t^2 + 50 - 5.9}{t - 3}$

$V(3) = \lim_{t \to 3} \frac{-4.9t^2 + 44.1}{t - 3}$

$V(3) = \lim_{t \to 3} \frac{-4.9(t^2 - 9)}{t - 3}$

$V(3) = \lim_{t \to 3} \frac{-4.9\cancel{(t-3)}(t + 3)}{\cancel{t-3}}$

$V(3) = \lim_{t \to 3} -4.9(t + 3) = -4.9(3 + 3) = -29.4$ m/s

Q4 a) $y = 4x - x^2, \; x = 1$

- Slope of the curve (i.e. slope of the tangent line)

$$f(1) = 4(1) - 1^2$$
$$f(1) = 4 - 1$$
$$f(1) = 3$$
$$(1, 3)$$

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$m_{tan} = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

$$m_{tan} = \lim_{x \to 1} \frac{4x - x^2 - 3}{x - 1}$$

$$m_{tan} = \lim_{x \to 1} \frac{-x^2 + 4x - 3}{x - 1}$$

$$m_{tan} = \lim_{x \to 1} \frac{-(x^2 - 4x + 3)}{x - 1}$$

$$m_{tan} = \lim_{x \to 1} \frac{-(x-3)\cancel{(x-1)}}{\cancel{x-1}}$$

$$m_{tan} = \lim_{x \to 1} -(x - 3)$$

$$m_{tan} = -(1-3) = -(-2) = 2$$

- equation of the tangent line

$(1, 3)$, $m_{tan} = 2$

$$y = mx + b \qquad\qquad y = 2x + 1$$
$$3 = 2(1) + b$$
$$3 - 2 = b$$
$$1 = b$$

- equation of the normal line

$$m_N = -\frac{1}{2} \quad (1, 3) \qquad y = mx + b \qquad\qquad y = \frac{1}{2}x + \frac{7}{2}$$
$$3 = -\frac{1}{2}(1) + b$$
$$3 + \frac{1}{2} = b$$
$$\frac{6 + 1}{2} = b$$
$$\frac{7}{2} = b$$

Q4 b) $y = x - x^3$, $x = -1$

- Slope of the curve

$$f(-1) = -1 - (-1)^3 = -1 - (-1) = -1 + 1 = 0$$

$$m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$m_{tan} = \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)}$$

$$m_{tan} = \lim_{x \to -1} \frac{x - x^3 - 0}{x + 1}$$

$$m_{tan} = \lim_{x \to -1} \frac{x(1 - x^2)}{x + 1}$$

$$m_{tan} = \lim_{x \to -1} \frac{x(1-x)\cancel{(1+x)}}{\cancel{x+1}}$$

$$m_{tan} = \lim_{x \to -1} x(1-x)$$

$$m_{tan} = -1(1-(-1)) = -1(2) = -2$$

- equation of tangent line

$(-1, 0)$, $m_t = -2$

$y = mx + b$

$0 = -2(-1) + b$

$-2 = b$

$y = -2x - 2$

- equation of normal line

$(-1, 0)$, $m_N = \frac{1}{2}$

$y = mx + b$

$0 = \frac{1}{2}(-1) + b$

$\frac{1}{2} = b$

$y = \frac{1}{2}x + \frac{1}{2}$

Q4. c) $y = \dfrac{2}{x+1}$, $x = -2$

• Slope of the curve

$f(-2) = \dfrac{2}{-2+1}$

$f(-2) = \dfrac{2}{-1}$

$f(-2) = -2$

$m_{tan} = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

$m_{tan} = \lim\limits_{x \to -2} \dfrac{f(x) - f(-2)}{x - (-2)}$

$m_{tan} = \lim\limits_{x \to -2} \dfrac{\frac{2}{x+1} - (-2)}{x+2}$

$m_{tan} = \lim\limits_{x \to -2} \dfrac{\frac{2}{x+1} + 2}{x+2}$

$m_{tan} = \lim\limits_{x \to -2} \dfrac{\left(\frac{2 + 2(x+1)}{x+1}\right)}{x+2}$

$m_{tan} = \lim\limits_{x \to -2} \dfrac{\frac{2 + 2x + 2}{x+1}}{x+2}$

$m_{tan} = \lim\limits_{x \to -2} \dfrac{\frac{2x + 4}{x+1}}{x+2}$

$m_{tan} = \lim\limits_{x \to -2} \dfrac{2\cancel{(x+2)}}{x+1} \cdot \dfrac{1}{\cancel{x+2}}$

$m_{tan} = \lim\limits_{x \to -2} \dfrac{2}{x+1}$

$m_{tan} = \dfrac{2}{-2+1}$

$m_{tan} = \dfrac{2}{-1}$

$m_{tan} = -2$

Q4 c) $y = \frac{2}{x+1}$, $x = -2$

• equation of the tangent line

$(-2,-2)$, $m_{tan} = -2$

$y = mx + b$

$-2 = -2(-2) + b$ $\qquad$ $y = -2x - 6$

$-2 = 4 + b$

$-6 = b$

• equation of the normal line

$(-2,-2)$, $m_N = \frac{1}{2}$

$y = mx + b$

$-2 = \frac{1}{2}(-2) + b$

$-2 = -1 + b$ $\qquad$ $y = \frac{1}{2}x - 1$

$-2 + 1 = b$

$-1 = b$

Q5: $y = t^2 - 8t \Rightarrow s(t) = t^2 - 8t$ $\quad$ $t = 2$

$V(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$ $\qquad$ $s(2) = 4 - 8(2)$

$s(2) = 4 - 16$

$s(2) = -12$

$V(2) = \lim_{t \to 2} \frac{s(t) - s(2)}{t - 2}$

$V(2) = \lim_{t \to 2} \frac{t^2 - 8t - (-12)}{t - 2}$

Q5 $V(2) = \lim_{t \to 2} \frac{t^2 - 8t + 12}{t-2}$

$V(2) = \lim_{t \to 2} \frac{(t-6)\cancel{(t-2)}}{\cancel{(t-2)}}$

$V(2) = \lim_{t \to 2} t - 6$

$V(2) = 2 - 6$

$V(2) = -4$ m/s

Q6. $A(x) = x^2, \quad x = 5$ cm

$\lim_{x \to a} \frac{A(x) - A(a)}{x - a}$ $\qquad A(5) = 25$

$= \lim_{x \to 5} \frac{x^2 - 25}{x - a}$

$= \lim_{x \to 5} \frac{\cancel{(x-5)}(x+5)}{\cancel{x-5}}$

$= \lim_{x \to 5} x + 5$

$= 5 + 5$

$= 10 \frac{cm^2}{cm}$

Q7 $y = \frac{x+1}{x-2}, \quad x = 2$

$x = 2$ is a non-permissibles value

There is a vertical asymptote at $x = 2$.

You cannot draw a tangent lines at $x = 2$.

Therefore, you cannot find the equation of the tangent lines.