AP Calculus BC Parametric, Polar, Vector Test Study Guide

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if possible, and find the slope and concavity (if possible) at the point corresponding to $t = 3$.

$$x = t + 1$$
$$y = t^2 + 6t$$

2. Find the arc length of the curve on the given interval. $x = t^2 + 1,\ y = 2t^3 + 1,\ 0 \le t \le 2$

3. Find the points of intersection of the graphs of the equations:

$$r = 1 + \cos\theta$$
$$r = 3\cos\theta$$

4. Find $\frac{dy}{dx}$ given:

$$x = t^2$$
$$y = 5 - 6t$$

5. Find the area enclosed by the polar curve $r\cos\left(\frac{1}{2}\theta\right) = 1$ in the interval $0 \le \theta \le \frac{\pi}{2}$.

6. The position vector for a particle moving in the *xy*-plane for $t \ge 0$ is $\langle\, 10\ln(1+t), 16\sqrt{t}\,\rangle$. What is the slope of the tangent line to the path of the particle at $t = 4$?

7. An object moving along a curve in the *xy*-plane is at position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \sin^{-1}\left(1 - 2e^{-t}\right) \text{ and } \frac{dy}{dt} = \frac{4t}{1 + t^3}$$

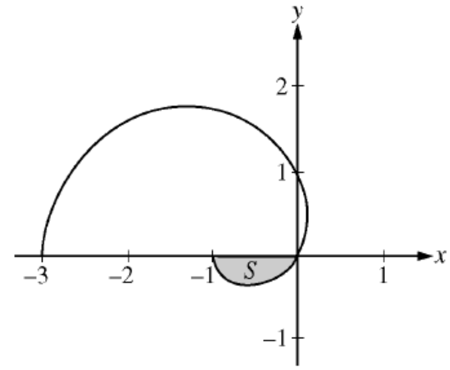
for $t \ge 0$. At time $t = 2$, the object is at the point $(6, -3)$. (Note: $\sin^{-1}x = \arcsin x$)

(a) Find the acceleration vector and the speed of the object at time $t = 2$.

(b) The curve has a vertical tangent line at one point. At what time t is the object at this point?

(c) Let $m(t)$ denote the slope of the line tangent to the curve at the point $(x(t), y(t))$. Write an expression for $m(t)$ in terms of t and use it to evaluate $\lim_{t\to\infty} m(t)$.

(d) The graph of the curve has a horizontal asymptote $y = c$. Write, but do not evaluate, an expression involving an improper integral that represents this value c.



8. The graph of the polar curve $r = 1 - 2\cos\theta$ for $0 \le \theta \le \pi$ is shown above. Let S be the shaded region in the third quadrant bounded by the curve and the x-axis.

(a) Write an integral expression for the area of S.

(b) Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

(c) Write an equation in terms of x and y for the line tangent to the graph of the polar curve at the point where $\theta = \frac{\pi}{2}$. Show the computations that lead to your answer.

9. A particle is moving along a curve so that its position at time t is $(x(t), y(t))$, where $x(t) = t^2 - 4t + 8$ and $y(t)$ is not explicitly given. Both x and y are measured in meters, and t is measured in seconds. It is known that $\frac{dy}{dt} = te^{t-3} - 1$.

(a) Find the speed of the particle at time $t = 3$ seconds.

(b) Find the total distance traveled by the particle for $0 \le t \le 4$ seconds.

(c) Find the time t, $0 \le t \le 4$, when the line tangent to the path of the particle is horizontal. Is the direction of motion of the particle toward the left or toward the right at that time? Give a reason for your answer.

(d) There is a point with x-coordinate 5 through which the particle passes twice. Find each of the following.

(i) The two values of t when that occurs

(ii) The slopes of the lines tangent to the particle's path at that point

(iii) The y-coordinate of that point, given $y(2) = 3 + \frac{1}{e}$

10. What integral represents the area inside one leaf of the polar rose $r = 3cos(2\theta)$?

A) $\int_0^{\frac{\pi}{4}} \sqrt{1+9\sin^2 2\theta}\, d\theta$

B) $\int_0^{\frac{\pi}{2}} \left(9\cos^2 2\theta\right) d\theta$

C) $\int_0^{\frac{\pi}{4}} \left(9\cos^2 2\theta\right) d\theta$

D) $\int_0^{\frac{\pi}{2}} \sqrt{1+9\sin^2 2\theta}\, d\theta$

E) $\frac{1}{2}\int_0^{\frac{\pi}{4}} \left(9\cos^2 2\theta\right) d\theta$

Answer Key

1. $\frac{dy}{dx} = 2t + 6,\ \frac{d^2y}{dx^2} = 2$. At $t = 3$: slope 12 and concave up
2. $\frac{2(37 \cdot \sqrt{37} - 1)}{27}$
3. $\left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, -\frac{\pi}{3}\right), (0,0)$
4. $\frac{dy}{dx} = -\frac{3}{t}$
5. 1
6. 2
7. (a) $a(2) = \langle 0.395 \text{ or } 0.396, -0.741 \text{ or } -0.740 \rangle$

   Speed $= \sqrt{x'(2)^2 + y'(2)^2} = 1.207$ or 1.208

   $2 : \begin{cases} 1 : \text{acceleration} \\ 1 : \text{speed} \end{cases}$

   (b) $\sin^{-1}\left(1 - 2e^{-t}\right) = 0$

   $1 - 2e^{-t} = 0$

   $t = \ln 2 = 0.693$ and $\frac{dy}{dt} \neq 0$ when $t = \ln 2$

   $2 : \begin{cases} 1 : x'(t) = 0 \\ 1 : \text{answer} \end{cases}$

   (c) $m(t) = \frac{4t}{1 + t^3} \cdot \frac{1}{\sin^{-1}\left(1 - 2e^{-t}\right)}$

   $$\lim_{t\to\infty} m(t) = \lim_{t\to\infty}\left(\frac{4t}{1+t^3} \cdot \frac{1}{\sin^{-1}\left(1 - 2e^{-t}\right)}\right) = 0\left(\frac{1}{\sin^{-1}(1)}\right) = 0$$

   $2 : \begin{cases} 1 : m(t) \\ 1 : \text{limit value} \end{cases}$

   (d) Since $\lim_{t\to\infty} x(t) = \infty$,

   $$c = \lim_{t\to\infty} y(t) = -3 + \int_2^\infty \frac{4t}{1+t^3}\, dt$$

   $3 : \begin{cases} 1\text{: integrand} \\ 1\text{: limits} \\ 1\text{: initial value consistent with lower limit} \end{cases}$

8.

(a) $r(0) = -1$; $r(\theta) = 0$ when $\theta = \frac{\pi}{3}$.

Area of $S = \frac{1}{2}\int_0^{\pi/3} (1 - 2\cos\theta)^2\, d\theta$

$2 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \end{cases}$

(b) $x = r\cos\theta$ and $y = r\sin\theta$

$$\frac{dr}{d\theta} = 2\sin\theta$$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta = 4\sin\theta\cos\theta - \sin\theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta = 2\sin^2\theta + (1 - 2\cos\theta)\cos\theta$$

$$4 : \begin{cases} 1 : \text{uses } x = r\cos\theta \text{ and } y = r\sin\theta \\ 1 : \dfrac{dr}{d\theta} \\ 2 : \text{answer} \end{cases}$$

(c) When $\theta = \frac{\pi}{2}$, we have $x = 0$, $y = 1$.

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\frac{\pi}{2}} = -2$$

The tangent line is given by $y = 1 - 2x$.

$$3 : \begin{cases} 1 : \text{values for } x \text{ and } y \\ 1 : \text{expression for } \dfrac{dy}{dx} \\ 1 : \text{tangent line equation} \end{cases}$$

9.

(a) Speed $= \sqrt{(x'(3))^2 + (y'(3))^2} = 2.828$ meters per second

1 : answer

(b) $x'(t) = 2t - 4$

Distance $= \int_0^4 \sqrt{(2t-4)^2 + \left(te^{t-3} - 1\right)^2}\, dt = 11.587$ or 11.588 meters

$$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$$

(c) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = 0$ when $te^{t-3} - 1 = 0$ and $2t - 4 \neq 0$

This occurs at $t = 2.20794$.

Since $x'(2.20794) > 0$, the particle is moving toward the right at time $t = 2.207$ or 2.208.

$$3 : \begin{cases} 1 : \text{considers } \dfrac{dy}{dx} = 0 \\ 1 : t = 2.207 \text{ or } 2.208 \\ 1 : \text{direction of motion with reason} \end{cases}$$

(d) $x(t) = 5$ at $t = 1$ and $t = 3$

At time $t = 1$, the slope is $\left.\dfrac{dy}{dx}\right|_{t=1} = \left.\dfrac{dy/dt}{dx/dt}\right|_{t=1} = 0.432$.

At time $t = 3$, the slope is $\left.\dfrac{dy}{dx}\right|_{t=3} = \left.\dfrac{dy/dt}{dx/dt}\right|_{t=3} = 1$.

$$y(1) = y(3) = 3 + \frac{1}{e} + \int_2^3 \frac{dy}{dt}\, dt = 4$$

$$3 : \begin{cases} 1 : t = 1 \text{ and } t = 3 \\ 1 : \text{slopes} \\ 1 : y\text{-coordinate} \end{cases}$$

10. C