Set 7: Multiple-Choice Questions on Applications of Integration to Geometry

Part A. Directions: Answer these questions *without* using your calculator.

AREA

In Questions 1–11, choose the alternative that gives the area of the region whose boundaries are given.

1. The curve of $y = x^2$, $y = 0$, $x = -1$, and $x = 2$.

(A) $\frac{11}{3}$ (B) $\frac{7}{3}$ (C) 3 (D) 5 (E) none of these

2. The parabola $y = x^2 - 3$ and the line $y = 1$.

(A) $\frac{8}{3}$ (B) 32 (C) $\frac{32}{3}$ (D) $\frac{16}{3}$ (E) none of these

3. The curve of $x = y^2 - 1$ and the y-axis.

(A) $\frac{4}{3}$ (B) $\frac{2}{3}$ (C) $\frac{8}{3}$ (D) $\frac{1}{2}$ (E) none of these

4. The parabola $y^2 = x$ and the line $x + y = 2$.

(A) $\frac{5}{2}$ (B) $\frac{3}{2}$ (C) $\frac{11}{6}$ (D) $\frac{9}{2}$ (E) $\frac{29}{6}$

5. The curve of $y = \frac{4}{x^2 + 4}$, the x-axis, and the vertical lines $x = -2$ and $x = 2$.

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{2}$ (C) 2π (D) π (E) none of these

6. The parabolas $x = y^2 - 5y$ and $x = 3y - y^2$.

(A) $\frac{32}{3}$ (B) $\frac{139}{6}$ (C) $\frac{64}{3}$ (D) $\frac{128}{3}$ (E) none of these

7. The curve of $y = \frac{2}{x}$ and $x + y = 3$.

(A) $\frac{1}{2} - 2\ln 2$ (B) $\frac{3}{2}$ (C) $\frac{1}{2} - \ln 4$

(D) $\frac{5}{2}$ (E) $\frac{3}{2} - \ln 4$

8. In the first quadrant, bounded below by the x-axis and above by the curves of $y = \sin x$ and $y = \cos x$.

(A) $2 - \sqrt{2}$ (B) $2 + \sqrt{2}$ (C) 2 (D) $\sqrt{2}$ (E) $2\sqrt{2}$

9. Bounded above by the curve $y = \sin x$ and below by $y = \cos x$ from $x = \frac{\pi}{4}$ to $x = \frac{5\pi}{4}$.

(A) $2\sqrt{2}$ (B) $\frac{2}{\sqrt{2}}$ (C) $\frac{1}{2\sqrt{2}}$

(D) $2(\sqrt{2} - 1)$ (E) $2(\sqrt{2} + 1)$

10. The curve $y = \cot x$, the line $x = \frac{\pi}{4}$, and the x-axis.

(A) $\ln 2$ (B) $\frac{1}{2} \ln \frac{1}{2}$ (C) 1 (D) $\frac{1}{2} \ln 2$ (E) 2

11. The curve of $y = x^3 - 2x^2 - 3x$ and the x-axis.

(A) $\frac{28}{3}$ (B) $\frac{79}{6}$ (C) $\frac{45}{4}$ (D) $\frac{71}{6}$ (E) none of these

12. The total area bounded by the cubic $x = y^3 - y$ and the line $x = 3y$ is equal to

(A) 4 (B) $\frac{16}{3}$ (C) 8 (D) $\frac{32}{3}$ (E) 16

13. The area bounded by $y = e^x$, $y = 1$, $y = 2$, and $x = 3$ is equal to

(A) $3 + \ln 2$ (B) $3 - 3 \ln 3$ (C) $4 + \ln 2$

(D) $3 - \frac{1}{2} \ln^2 2$ (E) $4 - \ln 4$

*14. The area enclosed by the ellipse with parametric equations $x = 2 \cos \theta$ and $y = 3 \sin \theta$ equals

(A) 6π (B) $\frac{9}{2}\pi$ (C) 3π (D) $\frac{3}{2}\pi$ (E) none of these

*An asterisk denotes a topic covered only in Calculus BC.

*15. The area enclosed by one loop of the cycloid with parametric equations $x = \theta - \sin\theta$ and $y = 1 - \cos\theta$ equals

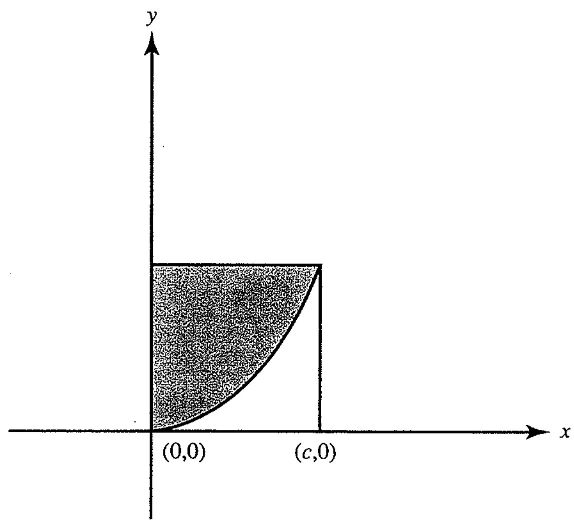
(A) $\frac{3\pi}{2}$ (B) 3π (C) 2π (D) 6π (E) none of these

16. The area enclosed by the curve $y^2 = x(1 - x)$ is given by

(A) $2\int_0^1 x\sqrt{1-x}\,dx$ (B) $2\int_0^1 \sqrt{x-x^2}\,dx$ (C) $4\int_0^1 \sqrt{x-x^2}\,dx$

(D) π (E) 2π

17. The figure below shows part of the curve of $y = x^3$ and a rectangle with two vertices at (0, 0) and (c, 0). What is the ratio of the area of the rectangle to the shaded part of it above the cubic?



(A) 3 : 4 (B) 5 : 4 (C) 4 : 3 (D) 3 : 1 (E) 2 : 1

VOLUME

In Questions 18–24 the region whose boundaries are given is rotated about the line indicated. Choose the alternative that gives the volume of the solid generated.

18. $y = x^2$, $x = 2$, and $y = 0$; about the x-axis.

(A) $\frac{64\pi}{3}$ (B) 8π (C) $\frac{8\pi}{3}$ (D) $\frac{128\pi}{5}$ (E) $\frac{32\pi}{5}$

19. $y = x^2$, $x = 2$, and $y = 0$; about the y-axis.

(A) $\frac{16\pi}{3}$ (B) 4π (C) $\frac{32\pi}{5}$ (D) 8π (E) $\frac{8\pi}{3}$

*An asterisk denotes a topic covered only in Calculus BC.

20. The first quadrant region bounded by $y = x^2$, the y-axis, and $y = 4$; about the y-axis.

(A) 8π **(B)** 4π **(C)** $\frac{64\pi}{3}$ **(D)** $\frac{32\pi}{3}$ **(E)** $\frac{16\pi}{3}$

21. $y = x^2$ and $y = 4$; about the x-axis.

(A) $\frac{64\pi}{5}$ **(B)** $\frac{512\pi}{15}$ **(C)** $\frac{256\pi}{5}$

(D) $\frac{128\pi}{5}$ **(E)** none of these

22. $y = x^2$ and $y = 4$; about the line $y = 4$.

(A) $\frac{256\pi}{15}$ **(B)** $\frac{256\pi}{5}$ **(C)** $\frac{512\pi}{5}$ **(D)** $\frac{512\pi}{15}$ **(E)** $\frac{64\pi}{3}$

23. An arch of $y = \sin x$ and the x-axis; about the x-axis.

(A) $\frac{\pi}{2}\left(\pi - \frac{1}{2}\right)$ **(B)** $\frac{\pi^2}{2}$ **(C)** $\frac{\pi^2}{4}$ **(D)** π^2 **(E)** $\pi(\pi - 1)$

24. A trapezoid with vertices at (2, 0), (2, 2), (4, 0), and (4, 4); about the x-axis.

(A) $\frac{56\pi}{3}$ **(B)** $\frac{128\pi}{3}$ **(C)** $\frac{92\pi}{3}$

(D) $\frac{112\pi}{3}$ **(E)** none of these

25. The base of a solid is a circle of radius a, and every plane section perpendicular to a diameter is a square. The solid has volume

(A) $\frac{8}{3}a^3$ **(B)** $2\pi a^3$ **(C)** $4\pi a^3$ **(D)** $\frac{16}{3}a^3$ **(E)** $\frac{8\pi}{3}a^3$

26. The base of a solid is the region bounded by the parabola $x^2 = 8y$ and the line $y = 4$, and each plane section perpendicular to the y-axis is an equilateral triangle. The volume of the solid is

(A) $\frac{64\sqrt{3}}{3}$ **(B)** $64\sqrt{3}$ **(C)** $32\sqrt{3}$

(D) 32 **(E)** none of these

27. The base of a solid is the region bounded by $y = e^{-x}$, the x-axis, the y-axis, and the line $x = 1$. Each cross section perpendicular to the x-axis is a square. The volume of the solid is

(A) $\frac{e^2}{2}$ **(B)** $e^2 - 1$ **(C)** $1 - \frac{1}{e^2}$

(D) $\frac{e^2 - 1}{2}$ **(E)** $\frac{1}{2}\left(1 - \frac{1}{e^2}\right)$

ARC LENGTH

***28.** The length of the arc of the curve $y^2 = x^3$ cut off by the line $x = 4$ is

(A) $\frac{4}{3}(10\sqrt{10} - 1)$ **(B)** $\frac{8}{27}(10^{3/2} - 1)$ **(C)** $\frac{16}{27}(10^{3/2} - 1)$

(D) $\frac{16}{27}10\sqrt{10}$ **(E)** none of these

***29.** The length of the arc of $y = \ln \cos x$ from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{3}$ equals

(A) $\ln\frac{\sqrt{3}+2}{\sqrt{2}+1}$ **(B)** 2 **(C)** $\ln(1 + \sqrt{3} - \sqrt{2})$

(D) $\sqrt{3} - 2$ **(E)** $\frac{\ln(\sqrt{3}+2)}{\ln(\sqrt{2}+1)}$

IMPROPER INTEGRALS

***30.** $\int_0^\infty e^{-x}\,dx =$

(A) 1 **(B)** $\frac{1}{e}$ **(C)** -1 **(D)** $-\frac{1}{e}$ **(E)** none of these

***31.** $\int_0^e \frac{du}{u} =$

(A) 1 **(B)** $\frac{1}{e}$ **(C)** $-\frac{1}{e^2}$ **(D)** -1 **(E)** none of these

***32.** $\int_1^2 \frac{dt}{\sqrt[3]{t-1}} =$

(A) $\frac{2}{3}$ **(B)** $\frac{3}{2}$ **(C)** 3 **(D)** 1 **(E)** none of these

*An asterisk denotes a topic covered only in Calculus BC.

***33.** $\int_2^4 \frac{dx}{(x-3)^{2/3}} =$

(A) 6 **(B)** $\frac{6}{5}$ **(C)** $\frac{2}{3}$ **(D)** 0 **(E)** none of these

***34.** $\int_2^4 \frac{dx}{(x-3)^2} =$

(A) 2 **(B)** –2 **(C)** 0 **(D)** $\frac{2}{3}$ **(E)** none of these

***35.** $\int_0^{\pi/2} \frac{\sin x}{\sqrt{1-\cos x}}\,dx$

(A) –2 **(B)** $\frac{2}{3}$ **(C)** 2 **(D)** $\frac{1}{2}$ **(E)** none of these

In Questions 36–40, choose the alternative that gives the area, if it exists, of the region described.

***36.** In the first quadrant under the curve of $y = e^{-x}$.

(A) 1 **(B)** e **(C)** $\frac{1}{e}$ **(D)** 2 **(E)** none of these

***37.** In the first quadrant under the curve of $y = xe^{-x^2}$.

(A) 2 **(B)** $\frac{2}{e}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{1}{2e}$ **(E)** none of these

***38.** In the first quadrant above $y = 1$, bounded by the y-axis and the curve $xy = 1$.

(A) 1 **(B)** 2 **(C)** $\frac{1}{2}$ **(D)** 4 **(E)** none of these

***39.** Between the curve $y = \frac{4}{1+x^2}$ and its asymptote.

(A) 2π **(B)** 4π **(C)** 8π **(D)** π **(E)** none of these

***40.** Between the curve $y = \frac{4}{\sqrt{1-x^2}}$ and its asymptotes.

(A) $\frac{\pi}{2}$ **(B)** π **(C)** 2π **(D)** 4π **(E)** none of these

*An asterisk denotes a topic covered only in Calculus BC.

In Questions 41 and 42, choose the alternative that gives the volume, if it exists, of the solid generated.

***41.** $y = \frac{1}{x}$, at the left by $x = 1$, and below by $y = 0$; about the x-axis.

(A) $\frac{\pi}{2}$ **(B)** π **(C)** 2π **(D)** 4π **(E)** none of these

***42.** The first-quadrant region under $y = e^{-x}$; about the x-axis.

(A) $\frac{\pi}{2}$ **(B)** π **(C)** 2π **(D)** 4π **(E)** none of these

Part B. Directions: Some of the following questions require the use of a graphing calculator.

AREA

In Questions 43–47, choose the alternative that gives the area of the region whose boundaries are given.

43. The area bounded by the parabola $y = 2 - x^2$ and the line $y = x - 4$ is given by

(A) $\int_{-2}^{3} (6 - x - x^2)\,dx$ **(B)** $\int_{-2}^{1} (2 + x + x^2)\,dx$ **(C)** $\int_{-3}^{2} (6 - x - x^2)\,dx$

(D) $2\int_{0}^{\sqrt{2}} (2 - x^2)\,dx + \int_{-3}^{2} (4 - x)\,dx$ **(E)** none of these

***44.** The area enclosed by the hypocycloid with parametric equations $x = \cos^3 t$ and $y = \sin^3 t$ is given by

(A) $3\int_{\pi/2}^{0} \sin^4 t \cos^2 t\,dt$ **(B)** $4\int_{0}^{1} \sin^3 t\,dt$ **(C)** $-4\int_{\pi/2}^{0} \sin^6 t\,dt$

(D) $12\int_{0}^{\pi/2} \sin^4 t \cos^2 t\,dt$ **(E)** none of these

45. Suppose the following is a table of ordinates for $y = f(x)$, given that f is continuous on $[1, 5]$:

x	1	2	3	4	5
y	1.62	4.15	7.5	9.0	12.13

If a trapezoid sum in used, with $n = 4$, then the area under the curve, from $x = 1$ to $x = 5$, is equal, to two decimal places, to

(A) 6.88 **(B)** 13.76 **(C)** 20.30 **(D)** 25.73 **(E)** 27.53

*An asterisk denotes a topic covered only in Calculus BC.

***46.** The area A enclosed by the four-leaved rose $r = \cos 2\theta$ equals, to three decimal places,

(A) 0.785 **(B)** 1.571 **(C)** 2.071 **(D)** 3.142 **(E)** 6.283

***47.** The area bounded by the small loop of the limaçon $r = 1 - 2\sin\theta$ is given by the definite integral

(A) $\int_{\pi/3}^{5\pi/3} \left[\frac{1}{2}(1-2\sin\theta)\right]^2 d\theta$

(B) $\int_{7\pi/6}^{3\pi/2} (1-2\sin\theta)^2\, d\theta$

(C) $\int_{\pi/6}^{\pi/2} (1-2\sin\theta)^2\, d\theta$

(D) $\int_{0}^{\pi/6} \left[\frac{1}{2}(1-2\sin\theta)\right]^2 d\theta + \int_{5\pi/6}^{\pi} \left[\frac{1}{2}(1-2\sin\theta)\right]^2 d\theta$

(E) $\int_{0}^{\pi/3} (1-2\sin\theta)^2\, d\theta$

VOLUME

In Questions 48–54 the region whose boundaries are given is rotated about the line indicated. Choose the alternative that gives the volume of the solid generated.

48. $y = x^2$ and $y = 4$; about the line $y = -1$.

(A) $4\pi \int_{-1}^{4} (y+1)\sqrt{y}\, dy$ **(B)** $2\pi \int_{0}^{2} (4-x^2)^2\, dx$ **(C)** $\pi \int_{-2}^{2} (16-x^4)\, dx$

(D) $2\pi \int_{0}^{2} (24-2x^2-x^4)\, dx$ **(E)** none of these

49. $y = 3x - x^2$ and $y = 0$; about the x-axis.

(A) $\pi \int_{0}^{3} (9x^2+x^4)\, dx$ **(B)** $\pi \int_{0}^{3} (3x-x^2)^2\, dx$ **(C)** $\pi \int_{0}^{\sqrt{3}} (3x-x^2)\, dx$

(D) $2\pi \int_{0}^{3} y\sqrt{9-4y}\, dy$ **(E)** $\pi \int_{0}^{9/4} y^2\, dy$

50. $y = 3x - x^2$ and $y = x$; about the x-axis.

(A) $\pi \int_{0}^{3/2} [(3x-x^2)^2 - x^2]\, dx$ **(B)** $\pi \int_{0}^{2} (9x^2-6x^3)\, dx$

(C) $\pi \int_{0}^{2} [(3x-x^2)^2 - x^2]\, dx$ **(D)** $\pi \int_{0}^{3} [(3x-x^2)^2 - x^4]\, dx$

(E) $\pi \int_{0}^{3} (2x-x^2)^2\, dx$

*An asterisk denotes a topic covered only in Calculus BC.

51. $y = \ln x$, $y = 0$, $x = e$; about the line $x = e$.

(A) $\pi \int_1^e (e - x) \ln x \, dx$ **(B)** $\pi \int_0^1 (e - e^y)^2 \, dy$ **(C)** $2\pi \int_1^e (e - \ln x) \, dx$

(D) $\pi \int_0^e (e^2 - 2e^{y+1} + e^{2y}) \, dy$ **(E)** none of these

***52.** The curve with parametric equations $x = \tan\theta$, $y = \cos^2\theta$, and the lines $x = 0$, $x = 1$, and $y = 0$; about the x-axis.

(A) $\pi \int_0^{\pi/4} \cos^4\theta \, d\theta$ **(B)** $\pi \int_0^{\pi/4} \cos^2\theta \sin\theta \, d\theta$ **(C)** $\pi \int_0^{\pi/4} \cos^2\theta \, d\theta$

(D) $\pi \int_0^1 \cos^2\theta \, d\theta$ **(E)** $\pi \int_0^1 \cos^4\theta \, d\theta$

53. A sphere of radius r is divided into two parts by a plane at distance h $(0 < h < r)$ from the center. The volume of the smaller part equals

(A) $\frac{\pi}{3}(2r^3 + h^3 - 3r^2h)$ **(B)** $\frac{\pi h}{3}(3r^2 - h^2)$ **(C)** $\frac{4}{3}\pi r^3 + \frac{h^3}{3} - r^2h$

(D) $\frac{\pi}{3}(2r^3 + 3r^2h - h^3)$ **(E)** none of these

54. If the curves of $f(x)$ and $g(x)$ intersect for $x = a$ and $x = b$ and if $f(x) > g(x) > 0$ for all x on (a, b), then the volume obtained when the region bounded by the curves is rotated about the x-axis is equal to

(A) $\pi \int_a^b f^2(x) \, dx - \int_a^b g^2(x) \, dx$

(B) $\pi \int_a^b [f(x) - g(x)]^2 \, dx$

(C) $2\pi \int_a^b x[f(x) - g(x)] \, dx$

(D) $\pi \int_a^b [f^2(x) - g^2(x)] \, dx$

(E) none of these

ARC LENGTH

***55.** The length of one arch of the cycloid $\begin{matrix} x = t - \sin t \\ y = 1 - \cos t \end{matrix}$ equals

(A) $\int_0^{\pi} \sqrt{1 - \cos t} \, dt$ **(B)** $\int_0^{2\pi} \sqrt{\frac{1 - \cos t}{2}} \, dt$ **(C)** $\int_0^{\pi} \sqrt{2 - 2\cos t} \, dt$

(D) $\int_0^{2\pi} \sqrt{2 - 2\cos t} \, dt$ **(E)** $2\int_0^{\pi} \sqrt{\frac{1 - \cos t}{2}} \, dt$

*An asterisk denotes a topic covered only in Calculus BC.

***56.** The length of the arc of the parabola $4x = y^2$ cut off by the line $x = 2$ is given by the integral

(A) $\int_{-1}^{1} \sqrt{x^2+1}\,dx$ **(B)** $\frac{1}{2}\int_{0}^{2} \sqrt{4+y^2}\,dy$ **(C)** $\int_{-1}^{1} \sqrt{1+x}\,dx$

(D) $\int_{0}^{2\sqrt{2}} \sqrt{4+y^2}\,dy$ **(E)** none of these

***57.** The length of $x = e^t \cos t$, $y = e^t \sin t$ from $t = 2$ to $t = 3$ is equal to

(A) $\sqrt{2}e^2\sqrt{e^2-1}$ **(B)** $\sqrt{2}(e^3-e^2)$ **(C)** $2(e^3-e^2)$

(D) $e^3(\cos 3 + \sin 3) - e^2(\cos 2 + \sin 2)$ **(E)** none of these

IMPROPER INTEGRALS

***58.** Which one of the following is an improper integral?

(A) $\int_{0}^{2} \frac{dx}{\sqrt{x+1}}$ **(B)** $\int_{-1}^{1} \frac{dx}{1+x^2}$ **(C)** $\int_{0}^{2} \frac{x\,dx}{1-x^2}$

(D) $\int_{0}^{\pi/3} \frac{\sin x\,dx}{\cos^2 x}$ **(E)** none of these

***59.** Which one of the following improper integrals diverges?

(A) $\int_{1}^{\infty} \frac{dx}{x^2}$ **(B)** $\int_{0}^{\infty} \frac{dx}{e^x}$ **(C)** $\int_{-1}^{1} \frac{dx}{\sqrt[3]{x}}$

(D) $\int_{-1}^{1} \frac{dx}{x^2}$ **(E)** none of these

***60.** Which one of the following improper integrals diverges?

(A) $\int_{0}^{\infty} \frac{dx}{1+x^2}$ **(B)** $\int_{0}^{1} \frac{dx}{x^{1/3}}$ **(C)** $\int_{0}^{\infty} \frac{dx}{x^3+1}$

(D) $\int_{0}^{\infty} \frac{dx}{e^x+2}$ **(E)** $\int_{1}^{\infty} \frac{dx}{x^{1/3}}$

*An asterisk denotes a topic covered only in Calculus BC.