CollegeBoard AP
connect to college success™

AP® Calculus:

Motion

2008
Curriculum Module

AP Calculus

Motion

Dixie Ross
Pflugerville High School
Pflugerville, TX

Motion along a straight line is an essential application of both differential and integral calculus and is a topic that appears often in AP Calculus Exam questions. Besides offering practical advice on how to help students understand the relationships between position, velocity, speed, and acceleration, this lesson encourages you to use the topic of motion to help students learn to reason from tables and graphs and communicate using correct language and notation.

Of all the types of free-response questions that have appeared on the AP Calculus Exam since it was first given in 1956, questions involving motion (relationships among position, velocity, and acceleration) are the most common type. For that reason, most teachers and their students are fairly comfortable with the basics of this topic and the performance, as judged by mean scores for these questions, is often quite good. That's why this topic provides a great opportunity to introduce students to questions based on numerical or graphical prompts and also to have students practice communicating their understanding of concepts using correct mathematical language and using correct mathematical notation in their work. Motion is also a topic that can be introduced fairly early in the school year in order to give students practice in answering free-response questions, and it can be revisited throughout the year as they acquire various skills and concepts. In this module, you will find instructional materials and practice problems that will be appropriate at three different times of the school year.

Introducing the Topic:

Once students have learned the basics of first and second derivatives, the topic of motion allows them to apply their skills in a meaningful context. To introduce students to the topic, I move back and forth along a number line taped off along the floor of the classroom or sometimes use a remote control car to represent the moving particle. As I discuss and illustrate the various concepts, students complete the fill-in-the-blanks Worksheet 1 for their notes. I will then give students a card with secret instructions (for example, "Your velocity should be positive and acceleration negative as you pass $x = 3$ on the number line.") and have the rest of the class guess as to what the instructions told them to do at $x = 3$. Students should then be ready to tackle a variety of practice problems—some numerical, some graphical, and some analytic. (See Worksheet 2.) Teachers should constantly stress the importance of correct vocabulary and notation.

Revisiting the Topic to Include the Concept of Speed:

Once students are comfortable with the initial concepts relating to the topic of motion and have successfully worked a variety of practice problems, it's time to revisit the topic later in the school year and discuss the concept of speed and how it relates to velocity and acceleration. In recent years, students have been asked on the free response questions of the AP Calculus Exam to determine if speed is increasing or decreasing at a particular point in time. Many students do not have a clear understanding of how to respond to this question in the most efficient way possible. Students can complete Worksheet 3, which includes looking at speed graphically and numerically so that they can learn how to correctly respond to these questions. The following day, they can complete the short quiz included with Worksheet 3 to assess their understanding.

Revisiting the Topic to Make Use of Integration:

Later in the year, after students have learned both the concepts and techniques of integration, we come back to the topic of motion one last time to complete our notes (Worksheet 4) and to practice various types of problems (Worksheet 5) before tackling a vast array of actual AP free-response questions.

At this point, teachers must make clear the distinctions among position, displacement, and distance traveled. Have a student (or the remote control car!) stand on the number line at $x = 2$, travel to the right to stand briefly on $x = 5$, and then travel to the left and come to a stop at $x = -1$.

Ask the class:

i. What is her position?

ii. What is her change in position (displacement)?

iii. What is her total distance traveled?

Answer:

i. $x = -1$

ii. -3 spaces

iii. 9 spaces (3 to the right and 6 to the left)

Repeat with similar examples until students seem able to keep these straight.

Worksheets and AP Examination Questions

Each of the worksheets includes additional notes for the instructor and complete solutions.

The chart below lists AP free-response questions that deal with concepts related to motion of a particle (though sometimes the particle takes the form of a car, a rocket, or an airplane). All of these questions, their solutions, and samples of student answers are readily available on AP Central® (apcentral.collegeboard.com) at The AP Calculus AB Exam page or at The AP Calculus BC Exam page. From the AP Calculus AB Course Home Page, select Exam Information: The AP Calculus AB Exam; from the AP Calculus BC Course Home Page, select Exam Information: The AP Calculus BC Exam.

Note that calculator-active questions require students to use their calculators, calculator-neutral questions are on the calculator-active portion of the exam but can easily be done without use of a calculator and calculator-inactive means the question appeared on the portion of the exam where the students do not have access to their calculators.

Year and Test Form	Question	Calculator Usage	Given Information
1997	AB 1	active	Velocity given as equation
1998	AB 3	neutral	Velocity of car given as data/graph
1999	AB 1	active	Velocity given as equation
2000	AB/BC 2	active	Velocity of runner A given as graph and velocity of runner B as equation
2001	AB/BC 3	neutral	Acceleration of car given as graph
2002	AB 3	active	Velocity given as equation
2002 Form B	AB 3	active	Velocity given as equation
2003	AB 2	active	Velocity given as equation
2003 Form B	AB 4	inactive	Velocity given as equation
2004	AB 3	active	Velocity given as equation
2004 Form B	AB/BC 3	active	Velocity of plane given as data/equation

Dixie Ross is a classroom teacher at Pflugerville High School, Pflugerville, Texas, and has over 23 years of teaching experience in all levels of mathematics from remedial to AP Calculus BC. She started the Advanced Placement program at Taylor High School in Taylor, Texas, in 1989, and began working as a College Board consultant and workshop leader in 1994. She served on the Development Committees for the Math Vertical Teams Toolkit, Setting the Cornerstones™ workshop, Building Success in Mathematics™ workshop, and AP Vertical Teams® Guide for Mathematics and Statistics, and has served as a Reader for the Advanced Placement Calculus Examination.

Worksheet 1. What You Need to Know About Motion Along the *x*-axis (Part 1)

In discussing motion, there are three closely related concepts that you need to keep straight. These are:

If $x(t)$ represents the position of a particle along the *x*-axis at any time *t*, then the following statements are true.

1. "Initially" means when ____________ = 0.
2. "At the origin" means ____________ = 0.
3. "At rest" means __________ = 0.
4. If the velocity of the particle is positive, then the particle is moving to the __________.
5. If the velocity of the particle is __________, then the particle is moving to the left.
6. To find average velocity over a time interval, divide the change in _______ by the change in time.
7. Instantaneous velocity is the velocity at a single moment (instant!) in time.
8. If the acceleration of the particle is positive, then the __________ is increasing.
9. If the acceleration of the particle is __________, then the velocity is decreasing.
10. In order for a particle to change direction, the __________ must change signs.
11. One way to determine total distance traveled over a time interval is to find the sum of the absolute values of the differences in position between all resting points. Here's an example: If the position of a particle is given by:

$$x(t) = \frac{1}{3}t^3 - t^2 - 3t + 4,$$

find the total distance traveled on the interval $0 \le t \le 6$.

Dixie Ross, Pflugerville High School, Pflugerville, Texas

Worksheet 1. Solutions and Notes for Students

The three concepts are as follows.

Position: $x(t)$—determines where the particle is located on the x-axis at a given time t

Velocity: $v(t) = x'(t)$—determines how fast the position is changing at a time t as well as the direction of movement

Acceleration: $a(t) = v'(t) = x''(t)$—determines how fast the velocity is changing at time t; the sign indicates if the velocity is increasing or decreasing

The true statements are as follows.

1. "Initially" means when __**time, *t***__ = 0.
2. "At the origin" means __**position, *x*(*t*)**__ = 0.
3. "At rest" means __**velocity, *v*(*t*)**__ = 0.
4. If the velocity of the particle is positive, then the particle is moving to the __**right**__.
5. If the velocity of the particle is __**negative**__, then the particle is moving to the left.
6. To find average velocity over a time interval, divide the change in __**position**__ by the change in time.
7. Instantaneous velocity is the velocity at a single moment (instant!) in time.
8. If the acceleration of the particle is positive, then the __**velocity**__ is increasing.
9. If the acceleration of the particle is __**negative**__, then the velocity is decreasing.
10. In order for a particle to change direction, the __**velocity**__ must change signs.
11. First, find the times at which $x'(t) = v(t) = 0$. That would be $t = -1$ (which is out of our interval) and $t = 3$. Next, evaluate the position at the end points and at each of the "resting" points. The particle moved to the left 9 units and then to the right by 27 units for a total distance traveled of 36 units. Point out to students how this is similar to a closed interval test, where you have to determine function values at the end points as well as at any critical points found.

t	$x(t)$
0	4
3	−5
6	22

Dixie Ross, Pflugerville High School, Pflugerville, Texas

Worksheet 2. Sample Practice Problems for the Topic of Motion (Part 1)

Example 1 (numerical).

The data in the table below give selected values for the velocity, in meters/minute, of a particle moving along the x-axis. The velocity v is a differentiable function of time t.

Time t (min)	0	2	5	6	8	12
Velocity $v(t)$ (meters/min)	-3	2	3	5	7	5

1. At $t = 0$, is the particle moving to the right or to the left? Explain your answer.

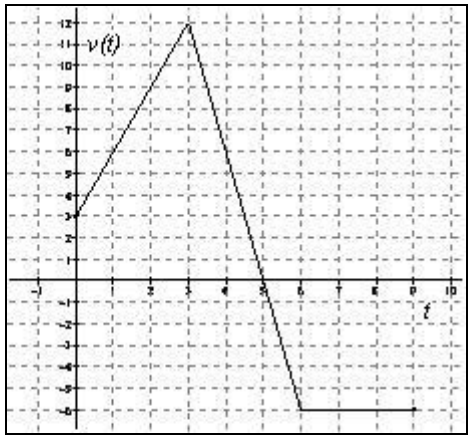
2. Is there a time during the time interval $0 \le t \le 12$ minutes when the particle is at rest? Explain your answer.

3. Use data from the table to find an approximation for $v'(10)$ and explain the meaning of $v'(10)$ in terms of the motion of the particle. Show the computations that lead to your answer and indicate units of measure.

4. Let $a(t)$ denote the acceleration of the particle at time t. Is there guaranteed to be a time $t = c$ in the interval $0 \le t \le 12$ such that $a(c) = 0$? Justify your answer.

Example 2 (graphical).

The graph below represents the velocity *v*, in feet per second, of a particle moving along the *x*-axis over the time interval from $t = 0$ to $t = 9$ seconds.



1. At $t = 4$ seconds, is the particle moving to the right or left? Explain your answer.

2. Over what time interval is the particle moving to the left? Explain your answer.

3. At $t = 4$ seconds, is the acceleration of the particle positive or negative? Explain your answer.

4. What is the average acceleration of the particle over the interval $2 \le t \le 4$? Show the computations that lead to your answer and indicate units of measure.

5. Is there guaranteed to be a time *t* in the interval $2 \le t \le 4$ such that $v'(t) = -3/2$ ft/sec^2? Justify your answer.

6. At what time t in the given interval is the particle farthest to the right? Explain your answer.

Example 3 (analytic).

A particle moves along the x-axis so that at time t its position is given by:

$$x(t) = t^3 - 6t^2 + 9t + 11$$

1. At $t = 0$, is the particle moving to the right or to the left? Explain your answer.

2. At $t = 1$, is the velocity of the particle increasing or decreasing? Explain your answer.

3. Find all values of t for which the particle is moving to the left.

4. Find the total distance traveled by the particle over the time interval $0 \le t \le 5$.

Dixie Ross, Pflugerville High School, Pflugerville, Texas

Worksheet 2. Solutions

Example 1 (numerical)

1. At $t = 0$, the particle is moving to the left because the velocity is negative.

2. Yes, there is a time when the particle is at rest during the time interval $0 < t < 12$ minutes. Since the velocity function is differentiable, it also is continuous. Hence, by the Intermediate Value Theorem, since velocity goes from negative to positive, it must go through zero and $v(t) = 0$ means the particle is at rest.

3. Since $t = 10$ is not one of the times given in the table, we should approximate the derivative by using a difference quotient with the best (closest) data available. Because 10 lies between 8 and 12, the best approximation is given by:

$$v'(10) \approx \frac{v(12)-v(8)}{12-8} = \frac{5-7}{12-8} = -\frac{1}{2}\ \frac{\text{m}/\text{min}}{\text{min}} = -\frac{1}{2}\frac{\text{m}}{\text{min}^2}.$$

   Here, $v'(10)$ is the acceleration of the particle at $t = 10$ minutes.

4. Yes, such a point is guaranteed by the Mean Value Theorem or Rolle's Theorem. Since velocity is differentiable (and therefore also continuous) over the interval $6 < t < 12$ and:

$$\frac{v(12)-v(6)}{12-6} = 0,$$

   then there must exist a point c in the interval such that $v'(c) = a(c) = 0$.

   Note: If we add the hypothesis that v' is continuous, then we may use the Intermediate Value Theorem to establish the result. In this case, since the values in the table indicate that velocity increases and then decreases on the interval $0 < t < 12$, then $v'(t) = a(t)$ must go from positive to negative and by the Intermediate Value Theorem must therefore pass through zero somewhere in that interval. It is the Mean Value Theorem, applied to the differentiable function v, that guarantees v' takes on at least one positive value in the interval $0 < t < 8$ (note that $\frac{7-(-3)}{8-0} = \frac{5}{4}$ is one such value), and at least one negative value in the interval $8 < t < 12$ (note that $\frac{5-7}{12-8} = -\frac{1}{2}$ is one such value).

Example 2 (graphical)

1. At $t = 4$ seconds, the particle is moving to the right because the velocity is positive.

2. The particle is moving to the left over the interval $5 < t \le 9$ seconds because the velocity is negative.

3. The acceleration of the particle is negative because the velocity is decreasing, OR the acceleration is the slope of the velocity graph and the slope of the velocity graph at $t = 4$ is negative.

4. Average acceleration over the time interval can be found by dividing the change in velocity by the change in time:

$$\frac{v(4)-v(2)}{4-2}=\frac{6-9}{4-2}=-\frac{3}{2}\frac{\text{ft}/\text{sec}}{\text{sec}}=-\frac{3}{2}\frac{\text{ft}}{\text{sec}^2}$$

5. No such time is guaranteed. The Mean Value Theorem does not apply since the function is not differentiable at $t = 3$ due to the sharp turn in the graph. If students have not yet learned the MVT, you can slide a tangent line (toothpick or stick) along the graph to show that no such point exists where the slope of the tangent line would be equal to the slope of a secant line between $t = 2$ and $t = 4$.

6. The particle is farthest to the right at $t = 5$ seconds. Since the velocity is positive during the time interval $0 \le t < 5$ seconds and negative during the time interval $5 < t \le 9$ seconds, the particle moves to the right before time $t = 5$ seconds and moves to the left after that time. Therefore, it is farthest to the right at $t = 5$ seconds.

Example 3 (analytic)

1. The particle is moving to the right because $x'(0) = v(0) = 9$ which is positive.

2. At $t = 1$, the velocity of the particle is decreasing because $x''(1) = v'(1) = a(1) = -6$, and if the acceleration is negative then the velocity is decreasing.

3. The particle is moving to the left for all values of t where $v(t) < 0$. We have:

$$v(t) = x'(t) = 3t^2 - 12t + 9 < 0 \text{ for } 1 < t < 3.$$

4.

t	$x(t)$
0	11
1	15
3	11
5	31

The particle moves right 4 spaces, left 4 spaces and then right 20 spaces. Therefore, the particle has traveled a total of 28 units. The common error that students make is to calculate $x(5) - x(0) = 20$, which gives the displacement or net change in position, rather than the total distance traveled. Teachers should reinforce the difference between these two concepts every chance they get.

Dixie Ross, Pflugerville High School, Pflugerville, Texas

Worksheet 3. Understanding the Relationships Among Velocity, Speed, and Acceleration

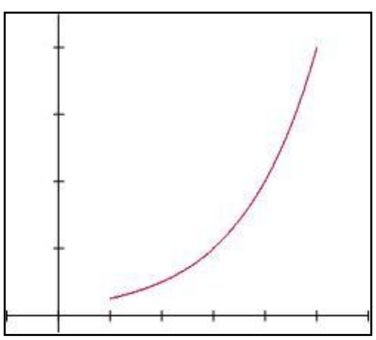
Speed is the absolute value of velocity. It tells you how fast something is moving without regard to the direction of movement.

1. What effect does absolute value have on numbers?

2. What effect does taking the absolute value of a function have on its graph?

For each situation below, the graph of a differentiable function giving velocity as a function of time t is shown for $1 \le t \le 5$, along with selected values of the velocity function. In the graph, each horizontal grid mark represents 1 unit of time and each vertical grid mark represents 4 units of velocity. For each situation, plot the speed graph on the same coordinate plane as the velocity graph and fill in the speed values in the table. Then, answer the questions below based on both the graph and the table of values.

Situation 1: Velocity graph



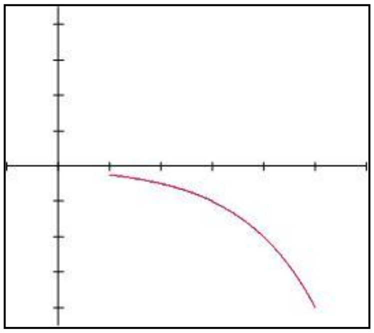
time	velocity	speed
1	1	
2	2	
3	4	
4	8	
5	16	

In this situation, the velocity is ______________ and ______________.
Positive or negative? Increasing or decreasing?

Because velocity is ______________, we know acceleration is ______________.
Increasing or decreasing? Positive or negative?

By examining the graph of speed and the table of values, we can conclude that speed is __________________.
Increasing or decreasing?

Situation 2: Velocity graph



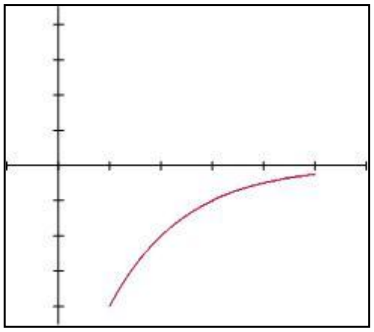
time	velocity	speed
1	-1	
2	-2	
3	-4	
4	-8	
5	-16	

In this situation, the velocity is _____________ and _____________.
Positive or negative? Increasing or decreasing?

Because velocity is ______________, we know acceleration is _____________.
Increasing or decreasing? Positive or negative?

By examining the graph of speed and the table of values, we can conclude that speed is _________________.
Increasing or decreasing?

Situation 3: Velocity graph



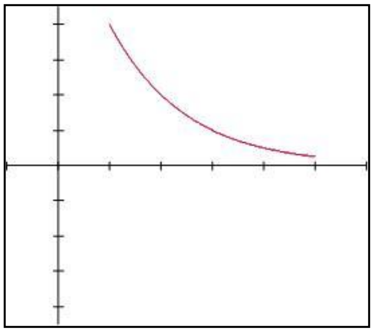
time	velocity	speed
1	-16	
2	-8	
3	-4	
4	-2	
5	-1	

In this situation, the velocity is _____________ and _____________.
Positive or negative? Increasing or decreasing?

Because velocity is ______________, we know acceleration is _____________.
Increasing or decreasing? Positive or negative?

By examining the graph of speed and the table of values, we can conclude that speed is _________________.
Increasing or decreasing?

Situation 4: Velocity graph



time	velocity	speed
1	16	
2	8	
3	4	
4	2	
5	1	

In this situation, the velocity is ______________ and ______________.
Positive or negative? Increasing or decreasing?

Because velocity is _______________, we know acceleration is ______________.
Increasing or decreasing? Positive or negative?

By examining the graph of speed and the table of values, we can conclude that speed is __________________.
Increasing or decreasing?

Conclusion:

In which situations was the speed increasing? ___________________________________

When the speed is increasing, the velocity and acceleration have ______________ signs.
Same or opposite?

In which situations was the speed decreasing? ___________________________________

When the speed is decreasing, the velocity and acceleration have ______________ signs.
Same or opposite?

Dixie Ross, Pflugerville High School, Pflugerville, Texas

Assessing Students' Understanding (A Short Quiz):

1. If velocity is negative and acceleration is positive, then speed is ____________.

2. If velocity is positive and speed is decreasing, then acceleration is __________.

3. If velocity is positive and decreasing, then speed is ______________________.

4. If speed is increasing and acceleration is negative, then velocity is __________.

5. If velocity is negative and increasing, then speed is ______________________.

6. If the particle is moving to the left and speed is decreasing, then acceleration is ____________________.

Dixie Ross, Pflugerville High School, Pflugerville, Texas

Worksheet 3. Solutions

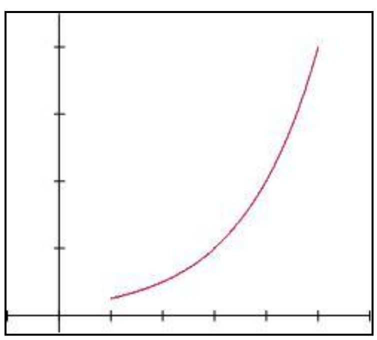
Speed is the absolute value of velocity. It tells you how fast something is moving without regard to the direction of movement.

1. What effect does absolute value have on numbers? **Absolute value makes all numbers non-negative.**

2. What effect does taking the absolute value of a function have on its graph? **Taking the absolute value of a function will cause any portion of its graph that is below the *x*-axis to be reflected across the *x*-axis.**

For each situation below, the graph of a differentiable function giving velocity as a function of time t is shown for $1 \le t \le 5$, along with selected values of the velocity function. In the graph, each horizontal grid mark represents 1 unit of time and each vertical grid mark represents 4 units of velocity. For each situation, plot the speed graph on the same coordinate plane as the velocity graph and fill in the speed values in the table. Then, answer the questions below based on both the graph and the table of values.

Situation 1: Speed graph



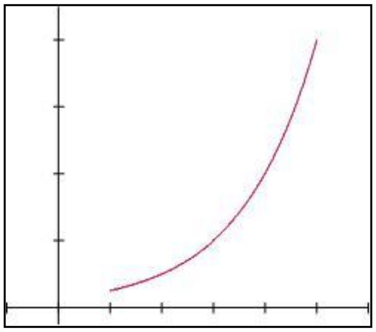
time	velocity	speed
1	1	**1**
2	2	**2**
3	4	**4**
4	8	**8**
5	16	**16**

In this situation, the velocity is ___***positive***___ and ___***increasing***___.
Positive or negative? Increasing or decreasing?

Because velocity is ___***increasing***___, we know acceleration is ___***positive***___.
Increasing or decreasing? Positive or negative?

By examining the graph of speed and the table of values, we can conclude that speed is ___***increasing***___.
Increasing or decreasing?

Situation 2: Speed graph



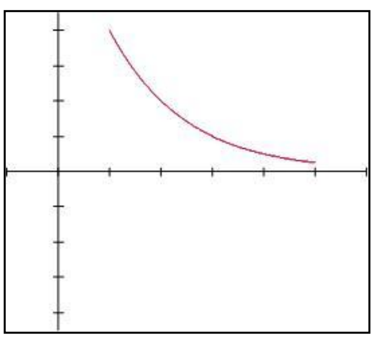
time	velocity	speed
1	-1	**1**
2	-2	**2**
3	-4	**4**
4	-8	**8**
5	-16	**16**

In this situation, the velocity is ___***negative***___ and ___***decreasing***___.
Positive or negative? Increasing or decreasing?

Because velocity is ___***decreasing***___, we know acceleration is ___***negative***___.
Increasing or decreasing? Positive or negative?

By examining the graph of speed and the table of values, we can conclude that speed is ___***increasing***___.
Increasing or decreasing?

Situation 3: Speed graph



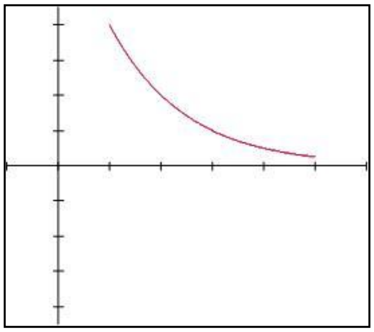
time	velocity	speed
1	-16	**16**
2	-8	**8**
3	-4	**4**
4	-2	**2**
5	-1	**1**

In this situation, the velocity is ___***negative***___ and ___***increasing***___.
Positive or negative? Increasing or decreasing?

Because velocity is ___***increasing***___, we know acceleration is ___***positive***___.
Increasing or decreasing? Positive or negative?

By examining the graph of speed and the table of values, we can conclude that speed is ___***decreasing***___.
Increasing or decreasing?

Situation 4: Speed graph



time	velocity	speed
1	16	**16**
2	8	**8**
3	4	**4**
4	2	**2**
5	1	**1**

In this situation, the velocity is ***positive*** and ***decreasing*** .
Positive or negative? Increasing or decreasing?

Because velocity is ***decreasing*** , we know acceleration is ***negative*** .
Increasing or decreasing? Positive or negative?

By examining the graph of speed and the table of values, we can conclude that speed is ***decreasing*** .
Increasing or decreasing?

Conclusion:

In which situations was the speed increasing? **Situations 1 and 2**

When the speed is increasing, the velocity and acceleration have **the same** signs.
Same or opposite?

In which situations was the speed decreasing? **Situations 3 and 4**

When the speed is decreasing, the velocity and acceleration have **opposite** signs.
Same or opposite?

Dixie Ross, Pflugerville High School, Pflugerville, Texas

Assessing Students' Understanding (A Short Quiz):

1. If velocity is negative and acceleration is positive, then speed is ***decreasing***.

2. If velocity is positive and speed is decreasing, then acceleration is ***negative***.

3. If velocity is positive and decreasing, then speed is ***decreasing***.

4. If speed is increasing and acceleration is negative, then velocity is ***negative***.

5. If velocity is negative and increasing, then speed is ***decreasing***.

6. If the particle is moving to the left and speed is decreasing, then acceleration is ***positive***.

Dixie Ross, Pflugerville High School, Pflugerville, Texas

Worksheet 4. What You Need to Know About Motion Along the *x*-axis (Part 2)

1. Speed is the absolute value of __________.

2. If the velocity and acceleration have the same sign (both positive or both negative), then speed is __________.

3. If the velocity and acceleration are __________ in sign (one is positive and the other is negative), then speed is decreasing.

There are three ways to use an integral in the study of motion that are easily confused. Watch out!

4. $\int v(t)\,dt$ is an _________ integral. It will give you an expression for _________ at time *t*. Don't forget that you will have a ______, the value of which can be determined if you know a position value at a particular time.

5. $\int_{t_1}^{t_2} v(t)\,dt$ is a ________ integral and so the answer will be a ________. The number represents the change in ________ over the time interval. By the Fundamental Theorem of Calculus, since $v(t) = x'(t)$, the integral will yield $x(t_2) - x(t_1)$. This is also known as displacement. The answer can be positive or _________ depending upon if the particle lands to the _____ or left of its original starting position.

6. $\int_{t_1}^{t_2} |v(t)|\,dt$ is also a _________ integral and so the answer will be a number. The number represents the _________________ traveled by the particle over the time interval. The answer should always be _____________.

Dixie Ross, Pflugerville High School, Pflugerville, Texas

Worksheet 4. Solutions

1. Speed is the absolute value of ___**velocity** ___.
2. If the velocity and acceleration have the same sign (both positive or both negative), then speed is ___ **increasing** ___.
3. If the velocity and acceleration are ___ **opposite** ___ in sign (one is positive and the other is negative), then speed is decreasing.

There are three ways to use an integral in the study of motion that are easily confused. Watch out!

4. $\int v(t)\,dt$ is an ___ **indefinite** ___ integral. It will give you an expression for ___ **position** ___ at time *t*. Don't forget that you will have a ___ **+ C, or constant of integration** ___, the value of which can be determined if you know a position value at a particular time.

5. $\int_{t_1}^{t_2} v(t)\,dt$ is a ___ **definite** ___ integral and so the answer will be a ___ **number** ___. The number represents the change in ___ **position** ___ over the time interval. By the Fundamental Theorem of Calculus, since $v(t) = x'(t)$, the integral will yield $x(t_2) - x(t_1)$. This is also known as displacement. The answer can be positive or ___ **negative** ___ depending upon if the particle lands to the ___ **right** ___ or left of its original starting position.

6. $\int_{t_1}^{t_2} |v(t)|\,dt$ is also a ___ **definite** ___ integral and so the answer will be a number. The number represents the ___ **total distance** ___ traveled by the particle over the time interval. The answer should always be ___ **non-negative** ___. *Note:* One way to see that:

$$\int_{t_1}^{t_2} |v(t)|\,dt$$

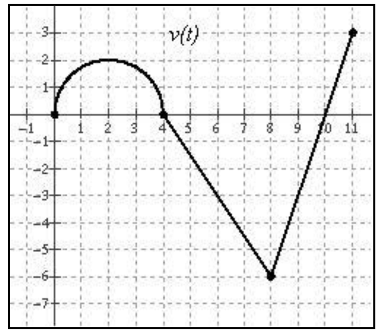
gives the total distance traveled is to note that the speed function $|v(t)|$, in transforming all negative velocities to positive velocities, also transforms all backward motion (motion to the left) to forward motion (motion to the right).
The change in position from start to finish of the resulting forward-only journey is the total distance traveled during the original forward-and-backward journey.

Dixie Ross, Pflugerville High School, Pflugerville, Texas

Worksheet 5. Sample Practice Problems for the Topic of Motion (Part 2)

Example 1 (graphical).

The graph to the right shows the velocity, $v(t)$, of a particle moving along the x-axis for $0 \leq t \leq 11$. It consists of a semicircle and two line segments. Use the graph and your knowledge of motion to answer the following questions.



1. At what time *t*, $0 \leq t \leq 11$, is the speed of the particle the greatest?

2. At which of the times, $t = 2$, $t = 6$ or $t = 9$, is the acceleration of the particle the greatest? Explain your answer.

3. Over what time intervals is the particle moving to the left? Explain your answer.

4. Over what time intervals is the speed of the particle decreasing? Explain your answer.

5. Find the total distance traveled by the particle over the time interval $0 \leq t \leq 11$.

6. Find the value of $\int_0^{11} v(t)\,dt$ and explain the meaning of this integral in the context of the problem.

7. If the initial position of the particle is $x(0) = 2$, find the position of the particle at time $t = 11$.

Example 2 (analytic/graphical/calculator active).

The rate of change, in kilometers per hour, of the altitude of a hot air balloon is given by $r(t) = t^3 - 4t^2 + 6$ for time $0 \le t \le 4$, where t is measured in hours. Assume the balloon is initially at ground level.

1. For what values of t, $0 \le t \le 4$, is the altitude of the balloon decreasing?

2. Find the value of $r'(2)$ and explain the meaning of the answer in the context of the problem. Indicate units of measure.

3. What is the altitude of the balloon when it is closest to the ground during the time interval $2 \le t \le 4$?

4. Find the value of $\int_0^4 r(t)\,dt$ and explain the meaning of the answer in the context of the problem. Indicate units of measure.

5. Find the value of $\int_0^4 |r(t)|\,dt$ and explain the meaning of the answer in the context of the problem. Indicate units of measure.

6. What is the maximum altitude of the balloon during the time interval $0 \le t \le 4$?

Example 3 (numerical).

The table below gives values for the velocity and acceleration of a particle moving along the x-axis for selected values of time t. Both velocity and acceleration are differentiable functions of time t. The velocity is decreasing for all values of t, $0 \le t \le 10$. Use the data in the table to answer the questions that follow.

Time, t	0	2	6	10
Velocity, $v(t)$	5	3	-1	-8
Acceleration, $a(t)$	0	-1	-3	-5

1. Is there a time t when the particle is at rest? Explain your answer.

2. At what time indicated in the table is the speed of the particle decreasing? Explain your answer.

3. Use a left Riemann sum to approximate $\int_0^{10} v(t)\,dt$. Show the computations you use to arrive at your answer. Explain the meaning of the definite integral in the context of the problem.

4. Is the approximation found in part (3) greater than or less than the actual value of the definite integral shown below? Explain your reasoning.

$$\int_0^{10} v(t)\,dt$$

5. Approximate the value of $\int_0^{10} |v(t)|\,dt$ using a trapezoidal approximation with the three sub-intervals indicated by the values in the table. Show the computations you use to arrive at your answer. Explain the meaning of the definite integral in the context of the problem.

6. Determine the value of $\int_0^{10} a(t)\,dt$. Explain the meaning of the definite integral in the context of the problem.

Dixie Ross, Pflugerville High School, Pflugerville, Texas

Worksheet 5. Solutions

Example 1 (graphical)

1. Speed is greatest at $t = 8$ since that is when the magnitude of the velocity is the greatest.
2. The acceleration is the derivative or slope of the velocity graph. Therefore:

$$a(2) = 0,\ a(6) = -\frac{3}{2}, \text{ and } a(9) = 3.$$

   The acceleration of the particle is greatest at $t = 9$.

3. The particle is moving to the left over the time interval $4 < t < 10$ because that is where the velocity is negative (velocity graph lies below the horizontal axis).
4. Whenever velocity and acceleration are opposite in sign, the speed is decreasing. The velocity is positive and decreasing for $2 < t < 4$, which means $v'(t) = a(t) < 0$ for $2 < t < 4$. This means the speed of the particle is decreasing for the time interval $2 < t < 4$ and in fact for the time interval $2 \le t \le 4$. The velocity is negative and increasing for $8 < t < 10$, which means $v'(t) = a(t) > 0$ for $8 < t < 10$. This means the speed of the particle also is decreasing for the time interval $8 < t < 10$ and in fact for the time interval $8 \le t \le 10$. Or, sketch and label clearly the graph of the speed function, $|v(t)|$, to see that the speed decreases for $2 \le t \le 4$ and $8 \le t \le 10$.
5. The total distance traveled by the particle is the sum of the areas of the regions between the velocity graph and the t-axis. Therefore the total distance traveled is given by the expression:

$$2\pi + 18 + \frac{3}{2} = 2\pi + \frac{39}{2}$$

   Equivalently, the total distance traveled is given by:

$$\int_0^{11} |v(t)|\,dt = 2\pi + 18 + \frac{3}{2} = 2\pi + \frac{39}{2}$$

6. The definite integral shown below represents the net change in position, or the displacement, of the particle over the time interval $0 \le t \le 11$.

$$\int_0^{11} v(t)\,dt = 2\pi - 18 + \frac{3}{2} = 2\pi - \frac{33}{2}$$

7. Using the Fundamental Theorem of Calculus,

$$x(11) = x(0) + \int_0^{11} v(t)\,dt = 2 + 2\pi - 18 + \frac{3}{2} = 2\pi - \frac{29}{2}$$

Example 2 (analytic/graphical/calculator active)

1. The altitude of the balloon is decreasing whenever $r(t) \le 0$. By graphing $r(t)$ and using the zero feature of the calculator, the time interval will be:

   $$1.571993 \le t \le 3.5141369 \text{ hours.}$$

   Students should be taught to store these values to use for subsequent calculations or to use more decimal places than the three that are required so that their subsequent answers will have three decimal place accuracy.

2. Students can use the numerical derivative feature of their calculator or compute by hand the value:

   $$r'(2) = -4 \ \frac{\text{km/hr}}{\text{hr}} = -4\frac{\text{km}}{\text{hr}^2}$$

   This value is the rate of change of the rate of change of the altitude of the hot air balloon at time $t = 2$ hours. In this case, at time $t = 2$ hours, the rate of change of the altitude is decreasing at a rate of 4 km/hr^2.

   Note: The rate of change of the altitude at time $t = 2$ hours is $r(2) = -2$ km/hr, indicating that the altitude itself also is decreasing at this time. Note also that if the balloon moved only vertically, then $r(t)$ would be its velocity and $r'(t)$ its acceleration. In this case, students could report that $r'(2) = -4$ means the acceleration of the balloon is -4 km/hr^2 or, equivalently, that the velocity of the balloon is decreasing at a rate of 4 km/hr^2.

3. By the result of part (1), the balloon is closest to the ground at $t = 3.514$ hours. If $y(t)$ denotes the altitude of the balloon after t hours, then, by the Fundamental Theorem of Calculus, the minimum altitude of the balloon for $2 \le t \le 4$ is:

   $$y(3.514) = y(0) + \int_0^{3.514} r(t)\,dt = 0 + 1.348 = 1.348 \text{ km.}$$

   Students should use the definite integral feature of the calculator to evaluate this integral.

4. The definite integral shown below represents the net change in the altitude of the balloon over the time interval $0 \le t \le 4$.

   $$\int_0^4 r(t)\,dt = 2.667 \text{ km.}$$

   Since the balloon was initially at ground level, this definite integral also gives the altitude of the balloon at $t = 4$ hours.

5. The definite integral shown below represents the total vertical distance traveled by the balloon over the time interval $0 \le t \le 4$. (Note that if the balloon moved only vertically, then $|r(t)|$ would be its speed.)

   $$\int_0^4 |r(t)|\,dt = 11.529 \text{ km.}$$

6. The maximum altitude of the balloon can occur either at the critical point where $r(t)$ goes from positive to negative ($t = 1.572$) or at the right endpoint of the time interval ($t = 4$). Since:

   $$y(1.572) = y(0) + \int_0^{1.572} r(t)\,dt = 0 + 5.779 = 5.779 \text{ km.}$$

   and since, by part (d), $y(4) = 2.667$ km, then the maximum altitude of the balloon is 5.779 km.

Example 3 (numerical)

1. Yes, there is a time when the particle is at rest. Since velocity is a differentiable function of time *t,* then velocity must also be continuous and the Intermediate Value Theorem can be applied. Since $v(2) = 3$ and $v(6) = -1$, then there must be a value of t in the interval $2 < t < 6$ such that $v(t) = 0$.

2. Speed is decreasing at $t = 2$ since at that time velocity and acceleration are opposite in sign.

3. Below, the definite integral represents the net change in position, or displacement, over the time interval $0 \le t \le 10$.

$$\int_0^{10} v(t)\,dt \approx 5\cdot 2 + 3\cdot 4 + (-1)(4) = 18$$

4. Because velocity is given to be a decreasing function on the interval $0 \le t \le 10$, the approximation in part (3) will be greater than the actual value of the integral.

5. The definite integral shown below represents the total distance traveled by the particle over the time interval $0 \le t \le 10$.

$$\int_0^{10} \left|v(t)\right| dt \approx \frac{1}{2}\cdot 2(5+3) + \frac{1}{2}\cdot 4(3+1) + \frac{1}{2}\cdot 4(1+8) = 34$$

6. Using the Fundamental Theorem of Calculus,

$$\int_0^{10} a(t)\,dt = v(10) - v(0) = -8 - 5 = -13$$

   The definite integral gives the net change in the velocity of the particle over the time interval $0 \le t \le 10$.

Dixie Ross, Pflugerville High School, Pflugerville, Texas