Day 4 Homework

Use your calculator on problems 10 and 13c only.

1. If $x = t^2 - 1$ and $y = e^{t^3}$, find $\dfrac{dy}{dx}$.

2. If a particle moves in the xy-plane so that at any time $t > 0$, its position vector is $\left\langle \ln\left(t^2 + 5t\right), 3t^2 \right\rangle$, find its velocity vector at time $t = 2$.

3. A particle moves in the xy-plane so that at any time t, its coordinates are given by $x = t^5 - 1$ and $y = 3t^4 - 2t^3$. Find its acceleration vector at $t = 1$.

4. If a particle moves in the xy-plane so that at time t its position vector is $\left\langle \sin\left(3t - \dfrac{\pi}{2}\right), 3t^2 \right\rangle$, find the velocity vector at time $t = \dfrac{\pi}{2}$.

5. A particle moves on the curve $y = \ln x$ so that its x-component has derivative $x'(t) = t + 1$ for $t \geq 0$. At time $t = 0$, the particle is at the point (1, 0). Find the position of the particle at time $t = 1$.

6. A particle moves in the xy-plane in such a way that its velocity vector is $\left\langle 1 + t, t^3 \right\rangle$. If the position vector at $t = 0$ is $\left\langle 5, 0 \right\rangle$, find the position of the particle at $t = 2$.

7. A particle moves along the curve $xy = 10$. If $x = 2$ and $\dfrac{dy}{dt} = 3$, what is the value of $\dfrac{dx}{dt}$?

8. The position of a particle moving in the xy-plane is given by the parametric equations $x = t^3 - \dfrac{3}{2}t^2 - 18t + 5$ and $y = t^3 - 6t^2 + 9t + 4$. For what value(s) of t is the particle at rest?

9. A curve C is defined by the parametric equations $x = t^3$ and $y = t^2 - 5t + 2$. Write the equation of the line tangent to the graph of C at the point (8, –4).

10. A particle moves in the xy-plane so that the position of the particle is given by $x(t) = 5t + 3\sin t$ and $y(t) = (8 - t)(1 - \cos t)$. Find the velocity vector at the time when the particle's horizontal position is $x = 25$.

11. The position of a particle at any time $t \geq 0$ is given by $x(t) = t^2 - 3$ and $y(t) = \dfrac{2}{3}t^3$.

(a) Find the magnitude of the velocity vector at time $t = 5$.

(b) Find the total distance traveled by the particle from $t = 0$ to $t = 5$.

(c) Find $\frac{dy}{dx}$ as a function of x.

12. Point $P(x, y)$ moves in the xy-plane in such a way that $\frac{dx}{dt} = \frac{1}{t+1}$ and $\frac{dy}{dt} = 2t$ for $t \geq 0$.

(a) Find the coordinates of P in terms of t given that, when $t = 1$, $x = \ln 2$ and $y = 0$.

(b) Write an equation expressing y in terms of x.

(c) Find the average rate of change of y with respect to x as t varies from 0 to 4.

(d) Find the instantaneous rate of change of y with respect to x when $t = 1$.

13. Consider the curve C given by the parametric equations $x = 2 - 3\cos t$ and $y = 3 + 2\sin t$, for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

(a) Find $\frac{dy}{dx}$ as a function of t.

(b) Find the equation of the tangent line at the point where $t = \frac{\pi}{4}$.

(c) The curve C intersects the y-axis twice. Approximate the length of the curve between the two y-intercepts.

Answers to Day 4 Homework

1. $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[e^{t^3}\right]}{\frac{d}{dt}\left[t^2 - 1\right]} = \frac{3t^2 e^{t^3}}{2t} = \frac{3te^{t^3}}{2}.$$

2. $$v(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle \frac{2t+5}{t^2+5t}, 6t \right\rangle \text{ so } v(2) = \left\langle \frac{9}{14}, 12 \right\rangle.$$

3. $$v(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle 5t^4, 12t^3 - 6t^2 \right\rangle.$$

$$a(t) = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle = \left\langle 20t^3, 36t^2 - 12t \right\rangle, \text{ so a}(1) = \langle 20, 24 \rangle.$$

4. $$v(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle 3\cos\left(3t - \frac{\pi}{2}\right), 6t \right\rangle \text{ so } v\left(\frac{\pi}{2}\right) = \langle 3\cos\pi, 3\pi \rangle = \langle -3, 3\pi \rangle.$$

5. $x(t) = \int (t+1)\,dt = \frac{t^2}{2} + t + C.$

$x(0) = 1 = C$ so $x(t) = \frac{t^2}{2} + t + 1.$

Since $x(1) = \frac{5}{2}$ and $y(1) = \ln\left(\frac{5}{2}\right)$, Position $= \left(\frac{5}{2}, \ln\left(\frac{5}{2}\right)\right).$

Or, since $x(1) = 1 + \int_0^1 (t+1)\,dt = \frac{5}{2}$ and $y(1) = \ln\left(\frac{5}{2}\right)$, Position $= \left(\frac{5}{2}, \ln\left(\frac{5}{2}\right)\right).$

6. $x(t) = \int (1+t)\,dt = t + \frac{t^2}{2} + C.$ $x(0) = 5$ so $C = 5$ and $x(t) = t + \frac{t^2}{2} + 5.$

$y(t) = \int t^3\,dt = \frac{t^4}{4} + D.$ $y(0) = 0$ so $D = 0$ and $y(t) = \frac{t^4}{4}.$

Position vector $= \left\langle t + \frac{t^2}{2} + 5, \frac{t^4}{4} \right\rangle.$ At $t = 2$, Position $= (9, 4).$

Or, since $x(2) = 5 + \int_0^2 (1+t)\,dt = 9$ and $y(2) = 0 + \int_0^2 t^3\,dt = 4,$

Position $= (9, 4).$

7. When $x = 2$, $y = 5$.

$x\frac{dy}{dt} + y\frac{dx}{dt} = 0$

$(2)(3) + 5\frac{dx}{dt} = 0$ so $\frac{dx}{dt} = -\frac{6}{5}$

Or find that $\frac{dy}{dx} = -\frac{10}{x^2}$. Then substituting into $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ gives $\frac{-10}{4} = \frac{3}{\frac{dx}{dt}}$ so that $\frac{dx}{dt} = -\frac{6}{5}$.

8. $x'(t) = 3t^2 - 3t - 18 = 3(t-3)(t+2) = 0$ when $t = 3$ and $t = -2$.

$y'(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3) = 0$ when $t = 3$ and $t = 1$.

The particle is at rest when $v(t) = \langle 0, 0 \rangle$ so at rest when $t = 3$.

9. $t^3 = 8$ $\qquad$ $t^2 - 5t + 2 = -4$

$t = 2$ $\qquad$ $t^2 - 5t + 6 = 0$

$(t-3)(t-2) = 0$

$t = 3,\ t = 2$

At $(8, -4)$ when $t = 2$

$$\left.\frac{dy}{dx}\right|_{t=2} = \left.\frac{2t-5}{3t^2}\right|_{t=2} = -\frac{1}{12}$$

Tangent line equation: $y + 4 = -\frac{1}{12}(x - 8)$

10. $5t + 3\sin t = 25$ when $t = 5.445755...$

$$v(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle 5 + 3\cos t, -1 + \cos t + (8 - t)\sin t \right\rangle$$

$$v(5.445755...) = \langle 7.008, -2.228 \rangle$$

11. (a) Magnitude when $t = 5$ is $\left.\sqrt{(2t)^2 + (2t^2)^2}\right|_{t=5} = \sqrt{2600}$ or $10\sqrt{26}$

(b)

$$\text{Distance } = \int_0^5 \sqrt{(2t)^2 + (2t^2)^2}\, dt = \int_0^5 2t\sqrt{1+t^2}\, dt = \left.\frac{2}{3}(1+t^2)^{\frac{3}{2}}\right|_0^5 = \frac{2}{3}\left(26^{\frac{3}{2}} - 1\right)$$

(c) $\frac{dy}{dx} = \frac{2t^2}{2t} = t = \sqrt{x+3}$

12. (a) $x(t)=\int \frac{1}{t+1}\,dt=\ln(t+1)+C.$ When $t=1,\ x=\ln 2$ so $C=0.$

$x(t)=\ln(t+1)$

$y(t)=\int 2t\,dt=t^2+D.$ When $t=1, y=0$ so $D=-1.$

$y(t)=t^2-1$

$(x,y)=\left(\ln(t+1),\ t^2-1\right)$

(b) $t+1=e^x$ so $t=e^x-1$ and $y=\left(e^x-1\right)^2-1=e^{2x}-2e^x.$

(c) Average rate of change $=\dfrac{y(b)-y(a)}{x(b)-x(a)}=\dfrac{y(4)-y(0)}{x(4)-x(0)}=\dfrac{15-(-1)}{\ln 5-\ln 1}=\dfrac{16}{\ln 5}$

(d) Instantaneous rate of change $=\left.\dfrac{dy}{dx}\right|_{t=1}=\left.\dfrac{2t}{\frac{1}{t+1}}\right|_{t=1}=4$

13. (a) $\dfrac{dy}{dx}=\dfrac{2\cos t}{3\sin t}=\dfrac{2}{3}\cot t$

(b) $y-\left(3+\sqrt{2}\right)=\dfrac{2}{3}\left(x-\left(2-\dfrac{3\sqrt{2}}{2}\right)\right)$

(c) $x=0$ when $t=-0.84106867\ldots,\ 0.84106867\ldots$

length $=\int_{-0.841\ldots}^{0.841,,,}\sqrt{(3\sin t)^2+(2\cos t)^2}\,dt=3.756$ or 3.757