Separable Differential Equations (4D)

The first order differential equation $F(y', y, x) = 0$ is **separable** if we can separate the variables x and y. Every separable differential equation can be written in a form

$$P(x) + Q(y)\frac{dy}{dx} = 0$$

or alternatively (if you solve for $\frac{dy}{dx}$ and rename the functions so that $p = -P$ and $q = 1/Q$), as

$$\frac{dy}{dx} = p(x)q(y)$$

To solve a separable differential equation,

- Rewrite the equation so that the left side has just one, and the right side just the other variable.
$$P(x)dx + Q(y)dy = 0 \quad \text{giving you} \quad P(x)dx = -Q(y)dy$$
- Integrate both sides.
- If possible, solve for the dependent variable.

Practice Problems.

1. Solve the differential equations and sketch the general solutions.
   a) $y' = 2y$
   b) $y'x = y$
   c) $y'y = -x$

   Then, solve the initial problems

   a) $y' = 2y, \quad y(0) = 3$
   b) $y'x = y, \quad y(2) = 4$
   c) $y'y = -x, \quad y(0) = 2$

2. Find the general solution of the following differential equations.
   a) $y' = xy$
   b) $y' = 3x^2y$
   c) $y' = (x + 1)y$

d) $y' = x(y+1)$

3. Find the solution of the differential equation that satisfies the given initial condition.

   a) $y' = \sqrt{4x+3}, \quad y(0) = 4/3$

   b) $y' = 2xy, \quad y(0) = 1$

   c) $y' = \frac{xy}{x^2+1}, \quad y(0) = 2$

(4E) 4. Free fall, no friction. A differential equation describing free fall with no friction is obtained by equating the total force with the opposite of the gravitational force. Thus,

$$m\frac{d^2x}{dt^2} = -mg$$

Find the function describing the height passed at time t if there is no initial velocity and the initial hight is x_0. Note that this is a second order differential equation. However, using that $v = dx/dt$, this can be reduced to two separable first order equations:

$$\frac{dv}{dt} = -g, \text{ with } v(0) = 0 \text{ and } \frac{dx}{dt} = v \text{ with } x(0) = x_0.$$

Note that here we treated a second order differential equation as a system of two first order differential equations. We shall later see that *every differential equation of order n can be reduced to a system of n first order differential equations.*

5. Exponential growth and decay. If the rate of growth of a quantity is proportional to the quantity size y at any time, this situation can be described by a differential equation

$$\frac{dy}{dt} = ky$$

The constant k is called the proportionality constant. If k is positive, the rate is positive (so that the quantity size is increasing) and if k is negative, the rate is negative (so that the quantity size is decreasing). Find the solution if $y(0) = y_0$.

Solutions.

1. a) Exponential functions $y = ce^{2x}$. b) Lines passing the origin $y = cx$ c) Circles centered at the origin for $c \geq 0$ $x^2 + y^2 = c$. No solutions for $c < 0$. For the second part of the problem a) Function $y = 3e^{2x}$. b) Line $y = 2x$. c) Circle $x^2 + y^2 = 4$.

2. a) $y = ce^{x^2/2}$ b) $y = ce^{x^3}$ c) $y = ce^{x^2/2+x}$ d) $y = ce^{x^2/2} - 1$

3. a) $y = 1/6 \cdot (4x+3)^{3/2} + 0.47$ b) $y = e^{x^2}$ c) $y = e^{1/2\ln(x^2+1)+\ln 2} = 2\sqrt{x^2+1}$

4. $x = x_0 - \frac{g}{2}t^2$.

5. $y = y_0 e^{kt}$.