The Indefinite Integral – Review (4A, 4B)

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if $F'(x) = f(x)$.

If $F(x)$ is an antiderivative of $f(x)$ then so is $F(x) + c$ for any constant c.

The **indefinite integral** of f is the antiderivative $F(x) + c$:

$$\int f(x)\, dx = F(x) + c$$

Rules of Integration:

1. Integrals of sums and differences: $\int (f(x) \pm g(x))\, dx = \int f(x)\, dx \pm \int g(x)\, dx$
2. Integrals of constant times a function: $\int k\ f(x)\, dx = k\ \int f(x)\, dx$
3. Power Rule:

$$\int x^n\, dx = \frac{1}{n+1}\ x^{n+1} + c$$

Practice Problems. Evaluate the integrals.

1. $\int (x^5 + 2)\, dx$ 2. $\int (4x^3 + 6x^2 - 4x + 3)\, dx$ 3. $\int \sqrt{x}\, dx$ 4. $\int \frac{3}{\sqrt{x}}\, dx$

5. $\int (\frac{1}{x^2} + x^2)\, dx$ 6. $\int (\sqrt{x} - \frac{4}{x^2})\, dx$ 7. $\int (2\sqrt[3]{x} + \frac{x^2}{4})\, dx$

8. Find the function $f(x)$ which has the derivative $f'(x) = \frac{1}{\sqrt[3]{x}}$ and satisfies the condition $f(8) = 9$.
9. Find the function $f(x)$ which has the derivative $f'(x) = 5\sqrt{x^3} + 3$ and satisfies the condition $f(1) = 4$.
10. Suppose that the velocity of an object is given by the function $v(t) = \frac{t}{2}$ where t is the time in seconds and v is the velocity in feet per second. Knowing that when $t = 2$ seconds, the position function $s(t) = 5$ feet, determine the position function $s(t)$.

Solutions.

1. $\frac{x^6}{6} + 2x + c$ 2. $x^4 + 2x^3 - 2x^2 + 3x + c$ 3. $\int \sqrt{x}\, dx = \int x^{1/2}\, dx = \frac{x^{3/2}}{3/2} = \frac{2}{3}\sqrt{x^3} + c$ 4. $\int \frac{3}{\sqrt{x}}\, dx = \int 3x^{-1/2}\, dx = 3\frac{x^{1/2}}{1/2} = 6\sqrt{x} + c$ 5. $\int (x^{-2} + x^2)\, dx = -\frac{1}{x} + \frac{x^3}{3} + c$

6. $\int (x^{1/2} - 4x^{-2})\, dx = \frac{2}{3}x^{3/2} + \frac{4}{x} + c$ 7. $\int (2x^{1/3} + \frac{1}{4}x^2)\, dx = 2\,\frac{3}{4}x^{4/3} + \frac{1}{4}\cdot\frac{1}{3}x^3 + c = \frac{3}{2}\sqrt[3]{x^4} + \frac{1}{12}x^3 + c$

8. $f(x) = \int f'(x)\, dx = \int x^{-1/3} dx = \frac{3}{2}\sqrt[3]{x^2} + c$. Using $f(8) = 9$ to solve for c, you have that $9 = \frac{3}{2}\sqrt[3]{8^2} + c = \frac{3}{2}\cdot 4 + c = 6 + c \Rightarrow 9 = 6 + c \Rightarrow c = 3$. Thus, $f(x) = \frac{3}{2}\sqrt[3]{x^2} + 3$.

9. $f(x) = \int f'(x)\, dx = \int (5x^{3/2} + 3)dx = 5\,\frac{2}{5}x^{5/2} + 3x + c = 2\sqrt{x^5} + 3x + c$. Using $f(1) = 4$ to solve for c, you have that $4 = 2\sqrt{1^5} + 3(1) + c = 2 + 3 + c = 5 + c \Rightarrow 4 = 5 + c \Rightarrow c = -1$. Thus, $f(x) = 2\sqrt{x^5} + 3x - 1$.

10. Recall that $s(t) = \int v(t)dt$. Thus $s(t) = \frac{1}{2}\,\frac{t^2}{2} + c = \frac{t^2}{4} + c$. Using that $s(2) = 5$, we have that $5 = \frac{2^2}{4} + c = 1 + c \Rightarrow c = 4$. Thus $s(t) = \frac{t^2}{4} + 4$.

Substitution – Review (4c)

If you cannot evaluate an integral directly using any of the formulas for finding the antiderivative, you can try the **substitution**. To use substitution you:

1. Choose the substitution u (usually the "inner" function: term under a radical, term in parenthesis, denominator, exponent...)
2. Compute du and solve for dx.
3. Substitute u and du. After the substitution, there should be no terms with x.
4. Integrate.
5. Substitute back.

Practice Problems. Evaluate the following.

1. $$\int (3x+5)^6 \; dx$$

2. $$\int (2x+1)^3 \; dx$$

3. $$\int \frac{x}{(x^2+3)^2} \; dx$$

4. $$\int \frac{x^2}{\sqrt{x^3-5}} \; dx$$

5. $$\int \frac{6}{\sqrt[3]{3x+5}} \; dx$$

6. Find the function $f(x)$ which has the derivative $f'(x) = \sqrt{2x+9}$ and satisfies the condition $f(0) = 5$.

7. Find the function $f(x)$ which has the derivative $f'(x) = \frac{10}{\sqrt{4x+1}}$ and satisfies the condition $f(0) = 3$.

8. Suppose that the velocity of an object is given by the function

$$v(t) = \frac{t}{\sqrt{t^2+9}}$$

where t is the time in seconds and v is the velocity in feet per second. Knowing that when $t = 4$ seconds, the position function $s(t) = 8$ feet, determine the position function $s(t)$.

Solutions.

1. Use substitution $u = 3x+5$. Then $du = 3dx$ so $dx = \frac{du}{3}$. $\int (3x+5)^6\, dx = \int u^6 \frac{du}{3} = \frac{1}{3}\int u^6 du = \frac{1}{3}\frac{u^7}{7} + c = \frac{u^7}{21} + c = \frac{(3x+5)^7}{21} + c$.

2. Use substitution $u = 2x+1$. Then $du = 2dx$ so $dx = \frac{du}{2}$. $\int (2x+1)^3\, dx = \int u^3 \frac{du}{2} = \frac{1}{2}\int u^3 du = \frac{1}{2}\frac{u^4}{4} + c = \frac{u^4}{8} + c = \frac{(2x+1)^4}{8} + c$.

3. Use substitution $u = x^2+3$. Then $du = 2xdx$ so $dx = \frac{du}{2x}$. $\int \frac{x}{(x^2+3)^2}\, dx = \int \frac{x}{u^2}\frac{du}{2x} = \frac{1}{2}\int u^{-2} du = \frac{1}{2}\frac{1}{-1}u^{-1} + c = \frac{-1}{2u} + c = \frac{-1}{2(x^2+3)} + c$.

4. Use substitution $u = x^3 - 5$. Then $du = 3x^2dx$ so $dx = \frac{du}{3x^2}$. $\int \frac{x^2}{\sqrt{x^3-5}}\, dx = \int \frac{x^2}{\sqrt{u}}\frac{du}{3x^2} = \frac{1}{3}\int u^{-1/2}du = \frac{1}{3}\frac{u^{1/2}}{1/2} + c = \frac{2\sqrt{u}}{3} + c = \frac{2\sqrt{x^3-5}}{3} + c$ or $\frac{2}{3}\sqrt{x^3-5} + c$.

5. Use substitution $u = 3x+5$. Then $du = 3dx$ so $dx = \frac{du}{3}$. $\int \frac{6}{\sqrt[3]{3x+5}}\, dx = 6\int \frac{1}{u^{1/3}}\frac{du}{3} = \frac{6}{3}\int u^{-1/3}du = 2\frac{u^{2/3}}{2/3} + c = 3u^{2/3} + c = 3(3x+5)^{2/3} + c$ or $3\sqrt[3]{(3x+5)^2} + c$.

6. $f(x) = \int f'(x)\, dx = \int \sqrt{2x+9}\, dx$. Use substitution $u = 2x+9$. Then $du = 2dx$ so $dx = \frac{du}{2}$. $\int \sqrt{2x+9}\, dx = \int u^{1/2}\frac{du}{2} = \frac{1}{2}\frac{u^{3/2}}{3/2} + c = \frac{1}{3}u^{3/2} + c = \frac{1}{3}(2x+9)^{3/2} + c$. Using $f(0) = 5$ to solve for c, you have that $5 = \frac{1}{3}\sqrt{9^3} + c = \frac{27}{3} + c \Rightarrow 5 = 9 + c \Rightarrow c = -4$. Thus, $f(x) = \frac{1}{3}(2x+9)^{3/2} - 4$.

7. $f(x) = \int f'(x)\, dx = \int \frac{10}{\sqrt{4x+1}}\, dx$. Use substitution $u = 4x+1$. Then $du = 4dx$ so $dx = \frac{du}{4}$. $\int \frac{10}{\sqrt{4x+1}}\, dx = 10\int u^{-1/2}\frac{du}{4} = \frac{10}{4}\frac{u^{1/2}}{1/2} + c = 5u^{1/2} + c = 5\sqrt{4x+1} + c$. Using $f(0) = 3$ to solve for c, you have that $3 = 5\sqrt{0+1} + c = 5 + c \Rightarrow 3 = 5 + c \Rightarrow c = -2$. Thus, $f(x) = 5\sqrt{4x+1} - 2$.

8. $s(t) = \int v(t)\, dt = \int \frac{t}{\sqrt{t^2+9}}\, dt$. Use substitution $u = t^2+9$. Then $du = 2tdt$ so $dt = \frac{du}{2t}$. $\int \frac{t}{\sqrt{t^2+9}}\, dt = \int \frac{t}{u^{1/2}}\frac{du}{2t} = \frac{1}{2}\int u^{-1/2}du = \frac{1}{2}\frac{u^{1/2}}{1/2} + c = u^{1/2} + c = \sqrt{t^2+9} + c$. Using $s(4) = 8$ to solve for c, you have that $8 = \sqrt{4+9} + c = 5 + c \Rightarrow 8 = 5 + c \Rightarrow c = 3$. Thus, $s(t) = \sqrt{t^2+9} + 3$.