MTH 148
Solutions for Problems on the Intermediate Value Theorem

1. Use the Intermediate Value Theorem to show that there is a positive number c such that $c^2 = 2$.

   Solution: Let $f(x) = x^2$. Then f is continuous and $f(0) = 0 < 2 < 4 = f(2)$. By the IVT there is $c \in (0, 2)$ such that $c^2 = f(c) = 2$.

2. If $f(x) = x^3 - x^2 + x$, show that there is $c \in \mathbb{R}$ such that $f(c) = 10$.

   Solution: $f(0) = 1$ and $f(3) = 3^3 - 3^2 + 3 = 27 - 9 + 3 = 21$, so $f(0) < 10 < f(3)$. Since f is continuous everywhere, there must be $c \in \mathbb{R}$ such that $f(c) = 10$.

3. If $g(x) = x^5 - 2x^3 + x^2 + 2$, show that there is $c \in \mathbb{R}$ such that $g(c) = -1$.

   Solution: First, g is continuous everywhere. Then $g(0) = 2$ and $g(-2) = -32 - 2(-8) + 4 + 2 = -32 + 22 = -10$. So $g(-2) < -1 < g(0)$. By IVT there is $c \in (-2, 0)$ such that $g(c) = -1$.

In problems 4–7, use the Intermediate Value Theorem to show that there is a root of the given equation in the given interval.

4. $x^3 - 3x + 1 = 0,\ (0, 1)$

   Solution: Let $f(x) = x^3 - 3x + 1$. Then f is continuous everywhere, $f(0) = 1$, and $f(1) = 1 - 3 + 1 = -1$. Therefore $f(1) < 0 < f(0)$ and by the IVT there is $x \in (0, 1)$ such that $f(x) = 0$.

5. $x^5 - 2x^4 - x - 3 = 0,\ (2, 3)$

   Solution: Let $f(x) = x^5 - 2x^4 - x - 3$. Then f is continuous everywhere, $f(2) = 32 - 32 - 2 - 3 = -5$, and $f(3) = 243 - 162 - 3 - 3 = 75$. Therefore $f(2) < 0 < f(3)$ and by the IVT there is $x \in (2, 3)$ such that $f(x) = 0$.

6. $x^3 + 2x = x^2 + 1,\ (0, 1)$

   Solution: Let $f(x) = x^3 - x^2 + 2x - 1$. Then f is continuous everywhere, $f(0) = -1$, and $f(1) = 1 - 1 + 2 - 1 = 1$. Therefore $f(0) < 0 < f(1)$ and by the IVT there is $x \in (0, 1)$ such that $f(x) = 0$ But then $x^3 - 2x^2 + 2x - 1 = 0$ or $x^3 + 2x = x^2 + 1$.

7. $x^2 = \sqrt{x+1}$, $(1,2)$

Solution: Let $f(x) = x^2 - \sqrt{x+1}$. Then f is continuous for all $x > -1$, $f(1) = 1 - \sqrt{2}$, and $f(2) = 4 - \sqrt{3}$. Therefore $f(1) < 0 < f(2)$ and by the IVT there is $x \in (1,2)$ such that $f(x) = 0$. But then $x^2 - \sqrt{x+1} = 0$ or $x^2 - \sqrt{x+1}$.

8. Let f be a continuous function on $[0,1]$. Show that if $-1 \le f(x) \le 1$ for all $x \in [0,1]$ then there is $c \in [0,1]$ such that $[f(c)]^2 = c$.

Solution: If $f(x)$ is continuous on $[0,1]$ then so is $[f(x)]^2$. Set $g(x) = [f(x)]^2 - x$. Then g is also continuous on $[0,1]$. Now $g(0) = [f(0)]^2 - 0 = [f(0)]^2 \ge 0$ and $g(1) = [f(1)]^2 - 1 \le 0$, so by IVT there is $c \in [0,1]$ such that $g(c) = 0$. Then $[f(c)]^2 - c = 0$ or $[f(c)]^2 = c$.