The Fundamental Theorem of Calculus (Part 1) Suppose that f is continuous on $[a,b]$. Then the function defined as $A(x) = \int\limits_a^x f(t)\,dt$ is continuous on $[a,b]$, differentiable on (a,b), and its derviative is $f(x)$:

$$\frac{d}{dx}\left(\int\limits_a^x f(t)\,dt\right) = f(x)$$

The Fundamental Theorem of Calculus (Part 2) Suppose that f is continuous on $[a,b]$ and $F' = f$ on $[a,b]$. Then

$$\int\limits_a^b f(x)\,dx = F(b) - F(a)$$

Sample Problems

1. Applications of Part 2: Compute each of the following definite integrals.

   a) $\int\limits_1^3 x^3 dx$ b) $\int\limits_0^5 \frac{1}{(x-3)^2} dx$ c) $\int\limits_0^1 \frac{1}{(x-3)^2} dx$

2. Applications of Part 1: Compute $\frac{dy}{dx}$ if

   a) $y = \int\limits_0^x t^2 dt$ c) $y = \int\limits_1^{x^3} t\sin(2t)\,dt$ e) $y = \left(\int\limits_0^x \cos t dt\right)^3$

   b) $y = \int\limits_x^5 \cos\left(m^2\right) dm$ d) $y = \int\limits_1^x \sqrt{t} dt - \int\limits_4^x \sqrt{t} dt$

Practice Problems

1. Compute each of the following definite integrals.

a) $\displaystyle\int_0^{\pi/4} \sin(2\theta)\,d\theta$ b) $\displaystyle\int_0^5 (2x+1)\,dx$ c) $\displaystyle\int_0^1 \frac{1}{1+x^2}dx$ d) $\displaystyle\int_1^3 \frac{1}{x^2}dx$ e) $\displaystyle\int_1^5 \sqrt{2x-1}dx$ f) $\displaystyle\int_0^{\pi/3} \sec\theta\tan\theta d\theta$ g) $\displaystyle\int_0^{\pi/4} \tan^2 y dy$

2. Compute the derivative $\dfrac{dy}{dx}$ if y is a function given as

a) $\displaystyle y=\int_0^x \cos^{-1} a\ da$

b) $\displaystyle y=\int_x^5 \sqrt{1+t^2}dt$

c) $\displaystyle y=\int_0^{x^2} \sqrt{1+t^2}dt$

d) $\displaystyle y=\int_0^{\cos x} \sqrt{1+t^2}dt$

e) $\displaystyle y=\int_0^x \tan\theta d\theta - \int_{\pi/4}^x \tan\theta d\theta$

f) $\displaystyle y=\int_0^{\sqrt{x}} \left(3t^2-4t+1\right)dt$

g) $\displaystyle y=\left(\int_0^x \left(6t^2-1\right)dt\right)^2$

Answers to Sample Problems

1. a) 20 b) undefined c) $\dfrac{1}{6}$

2. a) x^2 b) $-\cos\left(x^2\right)$ c) $3x^5\sin\left(2x^3\right)$ d) 0 e) $3\sin^2 x\cos x$

Answers to Practice Problems

1. a) $\dfrac{1}{2}$ b) 30 c) $\dfrac{\pi}{4}$ d) $\dfrac{2}{3}$ e) $\dfrac{26}{3}$ f) 1 g) $1-\dfrac{\pi}{4}$

2. a) $\dfrac{dy}{dx}=\cos^{-1}x$ b) $\dfrac{dy}{dx}=-\sqrt{1+x^2}$ c) $\dfrac{dy}{dx}=2x\sqrt{1+x^4}$ d) $\dfrac{dy}{dx}=-\sin x\sqrt{1+\cos^2 x}$
e) $\dfrac{dy}{dx}=0$ f) $\dfrac{dy}{dx}=\dfrac{1}{2\sqrt{x}}\left(3x-4\sqrt{x}+1\right)$ g) $\dfrac{dy}{dx}=2x\left(2x^2-1\right)\left(6x^2-1\right)$

Solutions of Sample Problems

1. a) $\displaystyle\int_1^3 x^3 dx$

Solution: In this case, $f(x) = x^3$. Clearly, $f(x) = x^3$ is continuous on $[1,3]$ and so the fundamental theorem can be applied. An antiderivative of f is $F(x) = \dfrac{x^4}{4}$. The (signed) area under the graph of f is the difference in the antiderivative F.

$$\int_1^3 x^3 dx = \frac{x^4}{4}\bigg|_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = \frac{80}{4} = \boxed{20}$$

b) $\displaystyle\int_0^5 \frac{1}{(x-3)^2} dx$

Solution: In this case, $f(x) = \dfrac{1}{(x-3)^2}$. Clearly, f is not continuous on $[0,5]$ because it has a discontinuity at $x = 3$. Consequently, the conditions of the fundamental theorem do not hold and so it can NOT be applied. For now, the correct solution of this integral is that it is undefined. We will later see methods developed addressing such an integral.

c) $\displaystyle\int_0^1 \frac{1}{(x-3)^2} dx$

Solution: In this case, $f(x) = \dfrac{1}{(x-3)^2}$. Clearly, $f(x)$ is continuous on $[0,1]$ and so the fundamental theorem can be applied. An antiderivative of f is $F(x) = -\dfrac{1}{x-3}$. The (signed) area under the graph of f is the difference in the antiderivative F.

$$\int_0^1 \frac{1}{(x-3)^2} dx = -\frac{1}{x-3}\bigg|_0^1 = -\frac{1}{1-3} - \left(-\frac{1}{0-3}\right) = \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}}$$

2. a) $y = \displaystyle\int_0^x t^2 dt$

Solution: We apply the fundamental theorem of calculus.

$$\frac{dy}{dx} = \frac{d}{dx}\int_0^x t^2 dt = \boxed{x^2}$$

b) $y = \int_x^5 \cos\left(m^2\right) dm$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}\int_x^5 \cos\left(m^2\right) dm = \frac{d}{dx}\left(-\int_5^x \cos\left(m^2\right) dm\right) = -\frac{d}{dx}\int_5^x \cos\left(m^2\right) dm = \boxed{-\cos\left(x^2\right)}$$

c) $y = \int_1^{x^3} t\sin(2t)\, dt$

Solution: The upper limit of integration is not x but x^3. This makes y a composite of two functions,

$$y = \int_1^u t\sin(2t)\, dt \quad \text{and} \quad u = x^3$$

To differentiate composed functions, we apply the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{d}{du}\left(\int_1^u t\sin(2t)\, dt\right)\cdot\frac{d}{dx}\left(x^3\right) = u\sin(2u)\cdot\left(3x^2\right) = x^3\sin\left(2x^3\right)\cdot\left(3x^2\right) = \boxed{3x^5\sin\left(2x^3\right)}$$

d) $y = \int_1^x \sqrt{t}dt - \int_4^x \sqrt{t}dt$

Solution:

$$\frac{d}{dx}\left(\int_1^x \sqrt{t}dt - \int_4^x \sqrt{t}dt\right) = \frac{d}{dx}\left(\int_1^x \sqrt{t}dt\right) - \frac{d}{dx}\left(\int_4^x \sqrt{t}dt\right) = \sqrt{x} - \sqrt{x} = \boxed{0}$$

This result is not all that surprising. If we apply properties of the definite integral, we see that y is constant and the derivative of a constant function is zero.

$$y(x) = \int_1^x \sqrt{t}dt - \int_4^x \sqrt{t}dt = \int_1^4 \sqrt{t}dt = \frac{2}{3}t^{3/2}\Big|_1^4 = \frac{2}{3}\left(4^{3/2} - 1^{3/2}\right) = \frac{2}{3}\cdot 7 = \frac{14}{3} \quad \text{and} \quad \frac{d}{dx}\left(\frac{14}{3}\right) = \boxed{0}$$

e) $y = \left(\int_0^x \cos t dt\right)^3$

Solution: This is a composed function, with

$$y = u^3 \quad \text{and} \quad u = \int_0^x \cos t dt$$

and so we will use the chain rule to differentiate it.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \cdot \frac{d}{dx}\left(\int_0^x \cos t dt\right) = 3\left(\int_0^x \cos t dt\right)^2 \cdot \cos x$$

We will compute the definite integral:

$$\int_0^x \cos t dt = \sin t \Big|_0^x = \sin x - \sin 0 = \sin x$$

and so the answer is

$$\frac{dy}{dx} = 3\left(\int_0^x \cos t dt\right)^2 \cdot \cos x = 3\left(\sin x\right)^2 \cdot \cos x = \boxed{3\sin^2 x \cos x}$$

For more documents like this, visit our page at http://www.teaching.martahidegkuti.com and click on Lecture Notes. E-mail questions or comments to mhidegkuti@ccc.edu.

 Last revised: January 21, 2014